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**Topological derivative method in control and design
of distributed parameter systems on metric graphs**

PhD dissertation

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Warszawa, 2024

Abstract

We consider the networks. The network is defined by a metric graph, $G = \{E, V\}$, and state equations on the set of edges E with some compatibility conditions at the vertices. There are applications of such models in transportation networks as well as for mechanical structures of beams. The state equations for dynamic problems are the wave equations on the graphs. The steady state equations are vectorial ordinary differential equations. The control problems on networks are considered for the system of wave equations. The optimum design problems for networks are considered for the steady state equations. The so-called *Turnpike Property* is shown to hold for the control problems with infinite horizon, which makes it possible to consider control and design bilevel optimization problems on networks. The cost for optimum design is the optimal cost for control.

This dissertation focuses on the development and application of the *Topological Derivative Method* in shape and topology optimization of networks. The singular perturbation of the tree graph is obtained, e.g., by nucleation of a small cycle. The size of the cycle is determined by the shape optimization technique in singularly perturbed domains. The small cycle changes the structure of the control system for the wave equation. That is why the small cycle is the singular perturbation of the control system on networks.

The original contribution of the dissertation includes among others:

- 1) The proof of Turnpike Property for abstract wave equations and the network. The results are obtained for the optimality system including the state equation, the adjoint state equation, and the sufficient optimality conditions for the wave equation and the steady state model.
- 2) The constructive form of topological derivative for complex networks including the network of Timoshenko beams with the cost for optimum design defined by the control problems with steady state equations.
- 3) The analysis of optimality conditions for nonlinear steady state equations in the form of local Pontryagin's maximum principle.
- 4) Numerical methods for solution of combined control and design problems on networks using Matlab with programs presented in Appendix.

Partial Differential Equations (PDEs) are considered on graphs, primarily of the tree structure. The singular perturbation of the shape is defined by a nucleation of a small cycle with the size $\varepsilon \rightarrow 0$. The cycle can be called a hole in the network. The location of the hole in the graph can be determined by the topological derivative method.

Optimal Control Problems (OCPs) are considered in the networks. The state equations defined on the graph are of evolution type, e.g., the wave equation for the networks. The steady state of the network is governed by specific Elliptic Boundary Value Problems (EBVPs).

Optimality conditions for optimal control problems arising in network modeling are derived. For the nonlinear state equations, we confine ourselves to the steady state network models. Therefore, we consider only control systems described by nonlinear ordinary differential equations. First, we derive optimality conditions for the nonlinear problem

for a single beam. These conditions are formulated in terms of the local Pontryagin maximum principle and the matrix Riccati equation. Then the optimality conditions for the control problem for networks posed on an arbitrary planar graph are discussed. Two simple numerical examples for the single-beam problem are considered.

The optimal control problems for linear state equations are introduced. The optimal distributed control problems for the Timoshenko beams are considered on networks. We employ the Steklov-Poincaré operator to convert the beam network model into a linear system of equations and utilize domain decomposition methods to enable topological changes. Specifically, the Timoshenko beams are known for their ability to model more complex mechanical behavior compared to simpler beam models. We present a method for computing such an optimality system and the topological derivative for tracking type of cost functionals, thereby enabling effective control and optimization of Timoshenko beam networks. Numerical examples validate our analytical results.

The optimal boundary control problems for the wave equation are considered on networks. Time interval $(0, T)$ for $T \rightarrow +\infty$ is fixed for the dynamic control problems. The so-called turnpike property is shown for the state equation, the adjoint state equation as well as the optimal cost. The shape and topology optimization is performed for the network with the shape functional given by the optimality system of control problem. The set of admissible shapes for the network is compact in finite dimensions, thus the use of turnpike property is possible. The topology optimization is analyzed for an example of nucleation of a small cycle at the internal node of network. The topological derivative of the cost is introduced and evaluated in the framework of domain decomposition technique. Numerical examples are provided.

We refer the reader to Lion's book [48] for optimal control theory, to Lewiński [44] for optimum design methods, to Novotny et al. [58, 57, 59] for the topological derivative method in optimum design.

List of Published Articles

1. M. Gugat, M. Qian, and J. Sokołowski. Topological derivative method for control of wave equation on networks. *The 27th International Conference on Methods and Models in Automation and Robotics (MMAR)*. IEEE, 2023. <https://doi.org/10.1109/MMAR58394.2023.10242484>
2. N. P. Osmolovskii, M. Qian, J. Sokołowski. Network optimality conditions. *Control and Cybernetics*, 2024. (accepted) <https://hal.science/hal-04279177>
3. M. Gugat, M. Qian, and J. Sokołowski. Network design and control: Shape and topology optimization for the turnpike property for wave equation, 2023. (submitted) <https://hal.science/hal-04285637>

In addition, the following papers are published:

4. M. Qian, S. Zhu. A level set method for Laplacian eigenvalue optimization subject to geometric constraints. *Computational Optimization and Applications*, 82(2), 499-524, 2022. <https://doi.org/10.1007/s10589-022-00371-1>
5. M. Qian, X. Hu., S. Zhu. A phase field method based on multi-level correction for eigenvalue topology optimization. *Computer Methods in Applied Mechanics and Engineering*, 401, 115646, 2022. <https://doi.org/10.1016/j.cma.2022.115646>
6. X. Hu, M. Qian, S. Zhu. Accelerating a phase field method by linearization for eigenfrequency topology optimization. *Structural and Multidisciplinary Optimization*, 66(12), 242, 2023. <https://doi.org/10.1007/s00158-023-03692-9>

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Chapter 1

Introduction

We consider optimal control problems and optimum design for networks. Such problems arise for example in gas and water transportation pipelines with compressors. Another important example could be the network of Timoshenko beams to model a bike. We are interested primarily in the dynamic control problems which can be reduced to the static control problems under appropriate conditions.

If we consider beams then we could have:

- Single load case and the elastic energy function;
- Multiply load case.

We use a more sophisticated model, where this cost for optimum design takes into account a set of parameters which are called controls. Therefore, the multiply load case is achieved by using the optimal control theory. The cost of design is determined by the optimality system for the control of networks. A specific choice is to define the design cost by the optimum value of control problem. For such a general shape functional we can use shape derivatives and topological derivatives for numerical solutions of design problems. This approach is new and efficient and combines the theory of optimal control of distributed parameters system with the modern technique of shape and topology optimization. We use also the so-called Turnpike Property of dynamic control problems with infinite horizons in order to use the steady state optimal problems for the numerical solutions of dynamic control problems.

The literature on control and optimization includes various foundational works and advancements. Lions' seminal contribution [48] is important in fundamental principles within optimal control theory. Furthermore, Leugering and Kogut's book [34] uses an optimality system of optimization, enriching our understanding of complex systems. In shape optimization, Sokołowski and Zolésio's book [80] is devoted to the shape sensitivity analysis for unilateral problems describing such physical phenomena as contact problems in elasticity, elasto-plastic torsion problems, and the obstacle problem. In topological derivatives, the works by Sokołowski and Żochowski [79], Sokołowski and Lewiński [45], and Sokołowski and Leugering [43] contribute significantly to understanding optimization

in structural design. Asymptotic analysis in singularly perturbed geometrical domains is needed. Moreover, Novotny’s books [58, 57, 59] recall the classical approach to shape optimization problems and extend the analysis to some singular perturbations of the reference domains for elliptic boundary value problems. The topological derivative can be considered as the singular limit of the classical shape derivative. In our dissertation, the optimal control theory is combined with the topological derivative method of shape optimization.

1.1 Motivation

In engineering, the optimum design of mechanical structures under specific loads is a popular task. To give a representative example, we refer e.g., to [77] for shape optimization approach in optimum design of prosthetic feet. Common objective functions encompass compliance, average stress, or structure volume [6], while the range of permissible shapes may be constrained by design limitations.

Typically, the optimization of a structure’s design or, more broadly, a domain involves an iterative process marked by incremental enhancements. Although shape optimization methods are effective for various modifications, including topology-altering changes like merging or splitting substructures, they are generally unsuitable for generating new subdomains [30], such as introducing holes in a structure (See Fig. 1.1). The strategic determination of where to create openings in the domain stands as a pivotal aspect of topology design [7, 20].

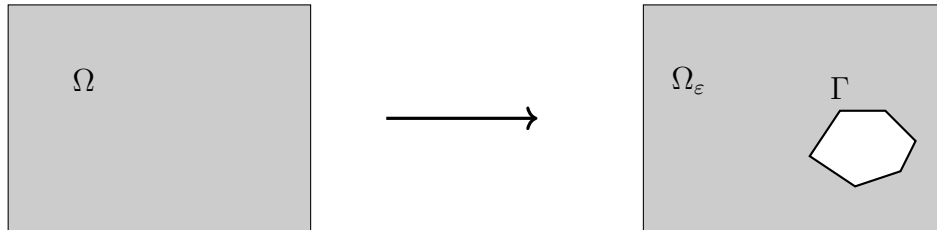


Figure 1.1: Topology change by creation of a hole (continuum).

Contrary to the suggestive nature of the term, the term “holes” within this domain does not necessarily denote emptiness. Depending on the context, these “holes” can signify material inclusions or variations [4, 76], concealed objects within a medium [10], or impediments influencing the flow of gases or liquids [13].

The utilization of topological derivatives extends to diverse scenarios, including optimal compliance problems [46, 78], shape functionals within the realm of linear elasticity [47, 57], as well as applications to semilinear elliptic systems [32], spectral problems [56], heat diffusion [57], and Helmholtz problems [73, 64].

Frequently, optimization methods integrate topological derivatives with level-set techniques to address the limitation of the latter, which prohibits the generation of new

subdomains [2, 3, 55, 83]. Overviews of the fields of use for topological gradients are given by [39] and [57].

The utilization of topological derivatives within beam networks revolves around optimizing the objective functional through deliberate vertex splitting, thereby inducing changes in the graph's configuration [19, 42, 43]. This process entails initially removing the targeted vertex and segments of its adjoining edges. Unlike scenarios in continuous domains where hole formations possess enclosed boundaries, the altered graph lacks such delineation around the void. Consequently, careful consideration is essential to define the desired structure post-reconnection following this modification (refer to Fig. 1.2). In Fig. 1.2, (a) is a Network, G is a star graph of the network, and G_ε is a graph in which the central node is replaced by a small cycle of size of ε .

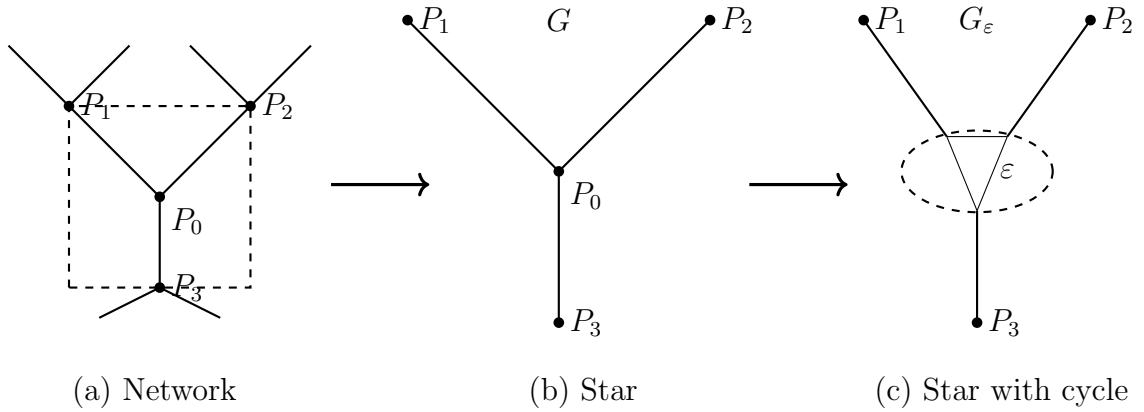


Figure 1.2: Topology change by creation of a small cycle .

Indeed, the optimum design of arrays of significant trusses and frames, as well as Michell Structures is studied in the monograph by Lewiński et al.[44]. Michell poses the query of safely and economically transmitting given loads to specified support zones without a priori assumptions on bar layout. The request for volume minimization transforms into maximizing virtual work, shifting the problem from trusses to optimal design of continuum bodies. Michell structures, optimal solutions to these problems, have inspired engineers and designers across disciplines, extending into modern fields like mechatronics and biotechnology in recent years.

1.2 Main Idea

We consider networks, which are described by partial differential equations (PDEs) on metric graphs. The first problem to solve is the optimal control problem. We consider the dynamic state equation, e.g., the wave equation. We perform shape and topology optimization of a cost functional, i.e., the shape functional. we are interested in problems with the turnpike property, which allows us to simplify numerical methods. In such a case, the dynamic problem could be simply replaced by a static problem, for which we

can use shape derivatives and topological derivatives for numerical solutions for the design of networks.

We consider state equations for Timoshenko beams and some other models of elastic structure for networks. We start with the description of shape optimization methods for networks.

The *boundary variations* technique in shape optimization is applied to the minimization of the functional

$$\Omega \mapsto J(\Omega), \quad (1.1)$$

which means the mapping from the geometrical domain to the scalar cost functional. As for applications, we will concentrate on equations of mathematical physics and mechanical engineering. This implies that Ω is representative of a physical body and, as a mathematical object, is the integration domain of a system of ordinary and more importantly PDEs describing the physical process of interest. In such a case, a composed mapping is considered. First

$$\Omega \mapsto y(\Omega) \quad (1.2)$$

is the solution of the state equation in the spatial domain, then the value of the cost function is given by $y(\Omega) \mapsto I(y(\Omega))$. Therefore, the composite mapping is given by

$$\Omega \mapsto y(\Omega) \mapsto I(y(\Omega)). \quad (1.3)$$

With (1.1) we associate shape gradients and the shape Hessians with respect to changes in Ω . The underlying state equations lead to corresponding material and shape derivatives for (1.2). Finally, with (1.3) we associate a so-called *adjoint state* equation to simplify the expression for the shape gradient of the composed mapping.

For regular perturbations of Ω , we use boundary variation techniques such that we restrict the analysis to the case of a family Ω_τ with $\tau \rightarrow 0$ and $\Omega_0 = \Omega$. This approach has its origin in fluid mechanics. As a result, a function $J(\cdot)$, originally defined as a function on domains is represented by a function of one variable in the vicinity of $\tau = 0^+$,

$$\tau \mapsto J(\Omega_\tau). \quad (1.4)$$

The case of singular domain perturbations is also investigated. By singular perturbations, we mean the introduction and evolution of holes or cracks. If a hole $B_\varepsilon(x)$ is present in Ω_ε , we consider a small parameter $\varepsilon \rightarrow 0^+$ and the shape functional being parametrized by

$$\varepsilon \mapsto J(\Omega_\varepsilon).$$

In such a case, an asymptotic analysis of the state equation is performed and the following formula is justified

$$J(\Omega_\varepsilon) = J(\Omega) + f(\varepsilon)\mathcal{T}_\Omega(x) + o(f(\varepsilon)),$$

with the function

$$x \rightarrow \mathcal{T}_\Omega(x)$$

denoted as the *topological derivative* of $J(\Omega)$ at $x \in \Omega$.

Two methods of shape sensitivity analysis can be used for shape and topology optimization. The first method is the so-called *velocity method* originated from the fluid mechanics [80]. The second is the so-called *topological derivative method* which is based on the singular perturbations of the geometrical domains [?, 59]. The first method replaces the mapping defined on the sets $\Omega \rightarrow y(x) := y(\Omega; x)$ by the mapping defined on the interval $(-\tau_0, \tau_0) \ni \tau \rightarrow y_\tau(x) := y(\Omega_\tau; x)$. Let us note that the function $x \rightarrow y_\tau(x)$ is defined on the set Ω_τ only. The second method uses the approximation of the solution $y(x \rightarrow \Omega_\varepsilon; x)$ in the domain with the small hole $\varepsilon \rightarrow \Omega_\varepsilon := \Omega \setminus \overline{\omega}_\varepsilon$ by the solution $x \rightarrow y(\Omega; x)$ in the intact domain Ω . The second method will be considered in particular for graphs in this dissertation.

1.3 Optimal Control Problems Governed by PDEs

In optimal control problems, we have a state equation and a tracking cost functional to be minimized. First, we derive the optimality system, which usually contains state equations, adjoint state equations, and optimality conditions. In the case of linear state equations, the associated optimality system admits a unique optimal control. In the case of nonlinear state equations, the solution of the optimality system is not unique. In the case of static problems of ODEs, Pontryagin's Maximum Principle is considered.

For the optimal value of cost, the shape and topological optimization of the network is studied. To this end, the shape and topological derivatives are introduced. We perform the design concerning the length of the edges. The topological derivatives are computed with respect to the nucleation of the small cycles.

1.4 Modeling, Control, and Design of Networks

The network structure is defined by a graph $G = (V, E)$, where V represents the set of nodes, E is the set of edges, and $n_v := \#V$, $n_e := \#E$. In the context of planar graphs, it is crucial to distinguish between one-dimensional “out-of-plane” evolution and two-dimensional “in-plane” evolution. This distinction necessitates the consideration of a function $y(x)$, which can be scalar, two-dimensional, or three-dimensional. In the latter two cases, *local coordinate* systems $(e_{ij})_{i=1, \dots, n_e}^{j=1, \dots, d}$, $d \in \{2, 3\}$ are required. For instance, in the 2D case, the expression for y on edge E_i takes the form:

$$y_i(x) = y_{1i}(x)e_i + y_{2i}(x)e_i^\perp,$$

where e_i denotes the local unit vector in the direction of the edge. So y_{1i} describes the longitudinal part of the displacement, while y_{2i} gives the lateral one.

In a network, we distinguish between single and multiple nodes. Multiple nodes inside the graph are called $v_M \in V_M \subset V$, while single nodes are $\partial V = V \setminus V_M$. They differ in

their boundary conditions. Dirichlet (clamped) nodes are permitted

$$v_D \in V_D := \{v_j \in \partial V \mid v_j \text{ is jammed} \} \subset V,$$

Neumann (free) nodes

$$v_N \in V_N := \{v_j \in \partial V \mid v_j \text{ is free} \} \subset V,$$

and controlled nodes

$$v_C \in V_C := \{v_j \in \partial V \mid v_j \text{ is controlled} \} \subset V.$$

We have $V = V_D \cup V_N \cup V_C \cup V_M$. Further, we define the index set ϵ_J as the set of indices for all edges adjacent to the node v_J , and d_J as the degree of the node v_J . So it is

$$V_M = \{v \in V \mid d(v) > 1\}.$$

Let L_i denote the length of the edge i and $\epsilon_{i,j}$ the orientation of the outer normal vector on the boundary (the nodes of the edge), i.e.,

$$\epsilon_{i,j} := \begin{cases} -1 & \text{if the } i\text{-th edge starts at node } v_j, \\ +1 & \text{if the } i\text{-th edge ends at node } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Figure 1.3 illustrates a single edge along with its local coordinate system. The coordinate system is centered at the node v_1 , with parameters $\epsilon_{i,1} = -1$ and $\epsilon_{i,2} = 1$ applied.

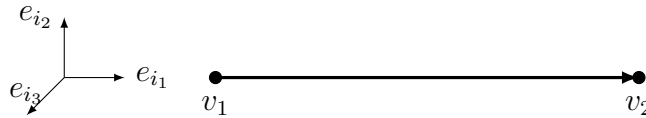


Figure 1.3: The local coordinate system of an edge.

The geometrical set associated with the graph is denoted by Ω . We consider a single edge, a cross of three edges, and a cross with a small cycle of six edges. If the system of PDEs is defined on the graph we can consider the shape sensitivity analysis of the state equation with respect to the domain perturbations of Ω . Thus we have the shape and material derivatives for regular perturbations and the topological derivatives for the singular perturbations. On edges, there are given PDEs, with the solutions y_i on E_i . For a single edge, the model of interest is a beam or wave equation.

1. *Timoshenko Beams model* (TB): A linear system of hyperbolic dynamic or elliptic static equations;
2. *Geometrically Exact Beams model* (GEB): Nonlinear beam;

3. *Intrinsic Geometrically Exact Beams model* (IGEB): A system of semilinear hyperbolic equations (weak formulation is used for approximation in space, then ODEs are solved);
4. Scalar wave equations.

In the vertices, there are two sorts of continuity conditions, the *continuity* for solutions and for fluxes which are the *Kirchhoff* conditions. They are also called *transmission* conditions. We consider the boundary optimal control problems for the second-order hyperbolic problems (wave equations) and distributed optimal control for the TB and static IGEB. The optimal value of the cost for control problems is subject to design with respect to the geometry

$$\Omega \rightarrow G = (E, V) \rightarrow y \rightarrow J(\Omega).$$

We evaluate the shape and the topological derivatives of the shape functional

$$\Omega \rightarrow J(\Omega).$$

The singular perturbation of the geometrical domain Ω means e.g. the nucleation of a small cycle of the size $\varepsilon \rightarrow 0$.

Remark In section 2.3.2 of [81] we have the Timoshenko beam model derived directly from GEB. We need the steady state models of beams (the models without time derivatives):

- from geometrically exact beams;
- from Timoshenko beams.

This way we could define two optimum design problems for steady state models:

- nonlinear beam or networks of beams by IGEB modeling;
- linear beam or networks of beams by Timoshenko modeling.

The solution of optimization problems will be tested if possible for time-dependent models of beams or networks.

The shape functional may be given by the optimal value of the cost for OCP in the case of a specific state equation. In such a case there are two techniques for solving the optimum design problems. The first strategy is Pontryagin's Maximum Principle for the total optimization problem i.e., control and design together. The second possibility is decomposition into the control problem at the lower level and the design problem at the upper level. In such a case, a unique optimal control is determined by an optimality system. In particular, the solution of an optimality system becomes simple in some cases with quadratic cost and without control and state constraints.

1.5 Principal Notations

Ω	a bounded open domain in $\mathbb{R}$
u	control
y	the state of the system
$SO(3)$	$\{\mathbf{S} \in \mathbb{R}^{3 \times 3} : \det(\mathbf{S}) = 1, \mathbf{S}^{-1} = \mathbf{S}^\top\}$
$C^k(\Omega)$	space of k -times continuously differentiable functions on $\bar{\Omega}$, k integer ≥ 0
$\mathcal{D}(\Omega)$	space of infinitely differentiable functions in Ω , with compact support in Ω , endowed with the inductive limit topology of Schwartz
$\mathcal{D}'(\Omega)$	dual space of $\mathcal{D}(\Omega)$, space of distributions on Ω
$L^2(\Omega)$	space (equivalence class) of functions square integrable on Ω
$H^m(\Omega)$	(Sobolev space of order m) space of functions φ such that $\varphi \in L^2(\Omega), \frac{\partial \varphi}{\partial x_i} \in L^2(\Omega), \dots, D^\alpha \varphi \in L^2(\Omega) \quad \forall \alpha, \alpha \leq m$
$H_0^m(\Omega)$	$\{\varphi \mid \varphi \in H^m(\Omega), D^\alpha \varphi = 0 \text{ on } \Gamma, \alpha \leq m-1\}$
$\mathcal{U}$	a Hilbert space on $\mathbb{R}$
$\mathcal{U}_{\text{ad}}$	(set of admissible controls) a closed, convex subset of $\mathcal{U}$
$\hat{u}$	is the skew-symmetric matrix defined by

$$\hat{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \text{ for any } u = (u_1, u_2, u_3)^\top \in \mathbb{R}^3$$

1.6 Outline and Main Results

We now give a brief description of the contents of the various chapters.

In Chapter 2, we present a comprehensive list of models under consideration, encompassing the Geometrically Exact Beams model and the Timoshenko Beams model, among others. Then, we study a detailed analysis of the characteristics of these models and their applications.

In Chapter 3, our analysis of the optimality conditions is performed for the nonlinear steady state models. Such an analysis can be useful for the real systems governed by the networks of Nonlinear Partial Differential Equations. The practical examples for our framework include e.g., the Gas and Hydrogen Distribution (GHD) Networks [25], [24], and the Geometrically Exact Beams (GEB) Networks which lead to the Intrinsic Geometrically Exact Beams (IGEB) network models [41], [70]. The GHD Networks are modeled by quasilinear hyperbolic systems. The IGEB Networks are governed by semi-linear hyperbolic systems under some assumptions on the transformation of GEB models. The steady state equations for two types of networks are given by ODEs.

In Chapter 4, we present the optimality system for both static and evolution optimization problems. We introduce the network model and provide the exact solution for both the 1D and 3D Timoshenko Beams models. Additionally, we employ the domain decomposition method to facilitate topology changes. Furthermore, we present numerical examples for both models to demonstrate their effectiveness and applicability in practical scenarios.

In Chapter 5, we consider two optimal control problems. The first problem, termed as *optimal control for evolution wave problem (OCE)*, is governed by an evolution equation. In order to define the second problem, denoted as *optimal control for static problem (OCS)*, the evolution state equation is reduced to the steady state equation. The optimal controls are given by the appropriate optimality systems, see [48]. In order to justify the approximation of (OCE) by (OCS), we study the turnpike property of the couple (OCE)-(OCS). For a recent survey on the turnpike property in optimal control, see [21] and the references therein. Several forms of the turnpike phenomenon have been studied in detail, for example, the exponential turnpike property (see [82]) and the interval turnpike property, see [18]. Numerical issues and the turnpike phenomenon in optimal shape design with parabolic PDEs have been studied in [38].

Numerical results are presented in Chapter 3,4,5. All programs in Matlab are in the Appendix. The numerical results confirm the theoretical results of the dissertation. The results of the dissertation are published in *Control and Cybernetics* (2024), a paper is submitted to the *Journal of Geometric Analysis* and the third paper is in preparation.

Chapter 2

Modeling of Beams

The first part of this dissertation is dedicated to deriving suitable descriptions of thin structures. These thin structures are commonly modeled as one-dimensional continua, i.e., the objects themselves and associated quantities are parameterized by only one spatial variable. All the models here are defined on a single edge.

2.1 Nonlinear Beams

The networks of elastic beams are of primal importance for applications we have in mind. Thus, we describe in detail the nonlinear models of beams that lead to semilinear state equations for static and evolution problems. The optimal steady state can be determined by solving the control problem for the static model. The practical examples for our framework include e.g., the *Geometrically Exact Beam* (GEB) networks which lead to the *Intrinsic Geometrically Exact Beam* (IGEB) network models [41], [70]. The IGEB networks are governed by semilinear hyperbolic systems under some assumptions on the transformation of GEB models. The steady state equations for two kinds of networks are given by ODEs. For convenience, we omit the index of edges.

2.1.1 Geometrically Exact Beam Model

The mathematical framework describing *geometrically exact beams* focuses on the position of the beam's centerline and the orientation of its cross sections with a fixed coordinate system. In the GEB context, the system state is denoted as $(\mathbf{p}, \mathbf{R})$. This state comprises the position of the centerline, denoted as $\mathbf{p}(x, t) \in \mathbb{R}^3$, and the orientation of the cross sections, represented by the columns $\{\mathbf{b}^j\}_{j=1}^3$ of the rotation matrix $\mathbf{R}(x, t) \in \text{SO}(3)$. For visual reference, we could refer to Fig. 2.1. The figure illustrates three pivotal states of a deformable beam: the unchanged reference beam; the initial beam characterized by a curvature described as $\Upsilon_c = \text{vec}\left(R^\top \frac{d}{dx} R\right)$, where $R = [b^1, b^2, b^3]$; and the beam at time t , represented by the state variables $\mathbf{p}$ and $\mathbf{R} = [\mathbf{b}^1, \mathbf{b}^2, \mathbf{b}^3]$.

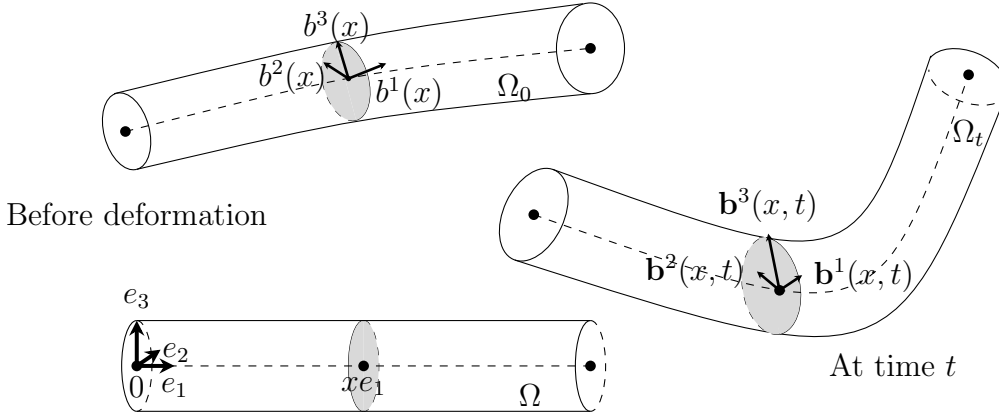


Figure 2.1: The straight reference beam (bottom), the beam before deformation (upper left), and the beam at time t (upper right).

For a beam with a length $L > 0$ positioned within the domain $(0, L) \times (0, T)$, the governing system is defined as follows:

$$\begin{bmatrix} \partial_t & \mathbf{0} \\ (\partial_t \hat{\mathbf{p}}) & \partial_t \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \mathbf{M} y_v = \begin{bmatrix} \partial_x & \mathbf{0} \\ (\partial_x \hat{\mathbf{p}}) & \partial_x \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} z + \begin{bmatrix} \bar{\phi} \\ \bar{\psi} \end{bmatrix}, \quad (2.1)$$

given external forces and moments $\bar{\phi}(x, t), \bar{\psi}(x, t) \in \mathbb{R}^3$, the mass matrix $\mathbf{M}(x) \in \mathbb{S}_{++}^6$ (the set of positive definite symmetric matrices), the flexibility (or compliance) matrix $\mathbf{C}(x) \in \mathbb{S}_{++}^6$ and the curvature before deformation $\Upsilon_c(x)$, and where y_v, z depend on $(\mathbf{p}, \mathbf{R})$:

$$y_v = \begin{bmatrix} \mathbf{R}^\top \partial_t \mathbf{p} \\ \text{vec}(\mathbf{R}^\top \partial_t \mathbf{R}) \end{bmatrix}, \quad s = \begin{bmatrix} \mathbf{R}^\top \partial_x \mathbf{p} - e_1 \\ \text{vec}(\mathbf{R}^\top \partial_x \mathbf{R}) - \Upsilon_c \end{bmatrix}, \quad z = \mathbf{C}^{-1} s. \quad (2.2)$$

and $y_v(x, t) \in \mathbb{R}^6$ represents linear and angular velocities, and $z(x, t) \in \mathbb{R}^6$ represents internal forces and moment.

2.1.2 Intrinsic Geometrically Exact Beams Model

Consider the *Intrinsic Geometrically Exact Beams* (IGEB) model for a single beam. The governing semilinear system consists of twelve equations. The state variable is denoted as

$$y = \begin{bmatrix} y_v \\ z \end{bmatrix}$$

expressed on a moving basis. That is, GEB model (2.1) and IGEB model (2.3) are related by the nonlinear transformation. We use y_v^f, z^f, y_v^l , and z^l to denote the first and last three components of y_v and z respectively. The notation $\bar{\Phi}(x, t)$ and $\bar{\Psi}(x, t) \in \mathbb{R}^3$ is employed for external forces and moments expressed in the moving basis. Within the

domain $(0, L) \times (0, T)$, the governing system of IGEB reads:

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \partial_t y - \begin{bmatrix} \mathbf{0} & \mathbb{I}_6 \\ \mathbb{I}_6 & \mathbf{0} \end{bmatrix} \partial_x y - \mathcal{A}y = -\mathcal{B}(y_v, z) \begin{bmatrix} \mathbf{M}y_v \\ \mathbf{C}z \end{bmatrix} + \begin{bmatrix} \bar{\Phi} \\ \bar{\Psi} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad (2.3)$$

where

$$\mathcal{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \hat{\Upsilon}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{e}_1 & \hat{\Upsilon}_c \\ \hat{\Upsilon}_c & \hat{e}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\Upsilon}_c & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathcal{B}(y_v, z) = \begin{bmatrix} \hat{y}_v^l & \mathbf{0} & \mathbf{0} & \hat{z}^f \\ \hat{y}_v^f & \hat{y}_v^l & \hat{z}^f & \hat{z}^l \\ \mathbf{0} & \mathbf{0} & \hat{y}_v^l & \hat{y}_v^f \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{y}_v^l \end{bmatrix}, \quad (2.4)$$

and $\mathbb{I}_6$ is the identity matrix with the size 6×6 . The system (2.3) is semilinear because of the presence on the right-hand side of the quadratic terms

$$(v, z) \mapsto \mathcal{B}(y_v, z) \begin{bmatrix} \mathbf{M}y_v \\ \mathbf{C}z \end{bmatrix}.$$

We introduce the matrix $\mathbf{E}(x) \in \mathbb{R}^{6 \times 6}$, which contains information about curvature and twist at rest, and the matrix $Q^{\mathcal{P}}(x) \in \mathbb{S}_{++}^{12}$, defined by

$$\mathbf{E} = \begin{bmatrix} \hat{\Upsilon}_c & \mathbf{0} \\ \hat{e}_1 & \hat{\Upsilon}_c \end{bmatrix}, \quad Q^{\mathcal{P}} = \text{diag}(\mathbf{M}, \mathbf{C}).$$

We present in a simple example, that of a single beam clamped at $x = 0$ and controlled via velocity free at $x = L$. The IGEB system with boundary conditions reads

$$\begin{cases} \partial_t y + \bar{A}(x) \partial_x y + \bar{B}(x) y = \bar{g}(x, y) & \text{in } (0, L) \times (0, T) \\ v(0, t) = 0 & \text{for } t \in (0, T) \\ z(l, t) = 0 & \text{for } t \in (0, T) \\ y(x, 0) = y^0(x) & \text{for } x \in (0, L) \end{cases} \quad (2.5)$$

where the coefficients $\bar{A}, \bar{B}$ and the source $\bar{g}$ depend on $\mathbf{M}, \mathbf{C}$ and $\mathbf{R}$, and $y^0(x)$ is the initial velocity. The governing system is derived by left-multiplying Eq. (2.3) by the inverse of $Q^{\mathcal{P}}$. Specifically, the functions $\bar{A}(x)$ and $\bar{B}(x)$ are defined over the interval $[0, L]$ and map to $\mathbb{R}^{12 \times 12}$,

$$\bar{A} = -(Q^{\mathcal{P}})^{-1} \begin{bmatrix} \mathbf{0} & \mathbb{I}_6 \\ \mathbb{I}_6 & \mathbf{0} \end{bmatrix}, \quad \bar{B} = (Q^{\mathcal{P}})^{-1} \begin{bmatrix} \mathbf{0} & -\mathbf{E} \\ \mathbf{E}^\top & \mathbf{0} \end{bmatrix}. \quad (2.6)$$

The function $\bar{g} : [0, L] \times \mathbb{R}^{12} \rightarrow \mathbb{R}^{12}$ is defined by

$$\bar{g}(x, u) = Q^{\mathcal{P}}(x)^{-1} \mathcal{G}(u) Q^{\mathcal{P}}(x) u$$

for all $x \in [0, L]$ and $u = (u_1^\top, u_2^\top, u_3^\top, u_4^\top)^\top \in \mathbb{R}^{12}$ with each $u_j \in \mathbb{R}^3$, where the map $\bar{\mathcal{G}}$ is defined by

$$\mathcal{G}(u) = - \begin{bmatrix} \hat{u}_2 & \mathbf{0} & \mathbf{0} & \hat{u}_3 \\ \hat{u}_1 & \hat{u}_2 & \hat{u}_3 & \hat{u}_4 \\ \mathbf{0} & \mathbf{0} & \hat{u}_2 & \hat{u}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{u}_2 \end{bmatrix}.$$

For the static problem, the nonlinear transformation results in $y_v = 0$. Denote $\mathbf{L}(z) := \begin{bmatrix} 0 & \hat{z}^f \\ \hat{z}^f & \hat{z}^l \end{bmatrix}$ and we have a steady state system:

$$\begin{cases} -\partial_x z = \mathbf{E}(x)z - \mathbf{L}(z)\mathbf{C}z + f(x)\mathbf{e}_1, & \text{in } (0, L) \\ z(L) = \mathbf{0}. \end{cases} \quad (2.7)$$

where $\partial_x = \frac{\partial}{\partial x}$, $f(x)$ is the distributed control and $\mathbf{e}_1 = (1, 0, 0, 0, 0, 0)^\top \in \mathbb{R}^6$.

2.2 Timoshenko Beams

The Timoshenko beam theory allows for cross-sections that are not perpendicular to the center line of the beam, and is more suitable for short or anisotropic beams [11]. It is also well-suited for problems with vibrating beams [31, 50]. More information about the modeling of Timoshenko beams can for example be found in [36, 37].

2.2.1 Notation and Parameters

One can prove the following representation for $\mathbb{E}_i$, the so-called Rodrigues formula

$$\begin{aligned} \mathbb{E}_i &= \mathbb{E}_i(\phi) = \cos \phi \mathbb{I} + \sin \phi \hat{\phi} e_i + (1 - \cos \phi) e_i e_i^\top \\ &\stackrel{\phi := \phi e_i}{=} \mathbb{I} + \frac{\sin \phi}{\phi} \hat{\phi} + \frac{1 - \cos \phi}{\phi^2} \hat{\phi}^2. \end{aligned} \quad (2.8)$$

For dynamic, cross sections' orientation by the columns $\{\mathbf{b}^j\}_{j=1}^3$ of $\mathbb{E}_i(x, t) \in \text{SO}(3)$. The displacements in the x -, y -, and z -directions are denoted by $r_{x,i}$, $r_{y,i}$, and $r_{z,i}$, correspondingly. These quantities are amalgamated into the displacement vector for steady state

$$r_i(x) := \mathbb{E}_i(\phi) (r_{x,i}(x), r_{y,i}(x), r_{z,i}(x))^\top$$

as well as for dynamic model

$$r_i(x, t) := \mathbb{E}_i(\phi_t) (r_{x,i}(x, t), r_{y,i}(x, t), r_{z,i}(x, t))^\top.$$

Similarly, the rotations in the x -, y -, and z -directions are represented by $\varphi_{x,i}$, $\varphi_{y,i}$, and $\varphi_{z,i}$, respectively, and combined into the rotation vector for steady state

$$\varphi_i(x) := \mathbb{E}_i(\phi) (\varphi_{x,i}(x), \varphi_{y,i}(x), \varphi_{z,i}(x))^\top$$

as well as for dynamic problem

$$\varphi_i(x, t) := \mathbb{E}_i(\boldsymbol{\phi}_t) (\varphi_{x,i}(x, t), \varphi_{y,i}(x, t), \varphi_{z,i}(x, t))^\top.$$

External loads, including distributed forces acting along the beam, are assumed to be piecewise continuous and can be summarized for steady state into the function

$$f_i \in \mathcal{PC}(0, L_i)^3, f_i(x) := \mathbb{E}_i(\boldsymbol{\phi}) (f_{x,i}(x), f_{y,i}(x), f_{z,i}(x))^\top,$$

as well as for dynamic problems

$$f_i \in L^1(\mathcal{PC}(0, L_i); (0, T))^3, f_i(x, t) := \mathbb{E}_i(\boldsymbol{\phi}_t) (f_{x,i}(x, t), f_{y,i}(x, t), f_{z,i}(x, t))^\top.$$

Similarly, distributed moments are combined for steady state into the vector function

$$m_i \in \mathcal{PC}(0, L_i)^3, m_i(x) := \mathbb{E}_i(\boldsymbol{\phi}) (m_{x,i}(x), m_{y,i}(x), m_{z,i}(x))^\top,$$

as well as for dynamic model

$$m_i \in L^1(\mathcal{PC}(0, L_i); (0, T))^3, m_i(x, t) := \mathbb{E}_i(\boldsymbol{\phi}_t) (m_{x,i}(x, t), m_{y,i}(x, t), m_{z,i}(x, t))^\top.$$

The combined displacement and rotation of a Timoshenko beam can be represented for steady state by six-dimensional vector functions as follows:

$$w_i(x) := \begin{pmatrix} r_i(x) \\ \varphi_i(x) \end{pmatrix} \quad \text{and} \quad q_i(x) := \begin{pmatrix} f_i(x) \\ m_i(x) \end{pmatrix}.$$

For the dynamic model, we use the notation:

$$w_i(x, t) := \begin{pmatrix} r_i(x, t) \\ \varphi_i(x, t) \end{pmatrix} \quad \text{and} \quad q_i(x, t) := \begin{pmatrix} f_i(x, t) \\ m_i(x, t) \end{pmatrix}.$$

These vector functions will be extensively used in the subsequent sections of this study. We list the notation and parameters that will be used in our model in Table 2.1. The coefficients k_i^x , k_i^s , c_i^x , c_i^y , and c_i^z are the stiffness and damping coefficients, which are beam-specific and depend on the material properties and geometry of the beam. The internal forces $F_i^x, F_i^y, F_i^z : [0, L_i] \rightarrow \mathbb{R}^3$ of a Timoshenko beam are given by

$$\begin{aligned} F_i^x(x) &:= k_i^x r'_{x,i}(x), \\ F_i^y(x) &:= k_i^s (r'_{y,i}(x) - \varphi_{z,i}(x)), \\ F_i^z(x) &:= k_i^s (r'_{z,i}(x) + \varphi_{y,i}(x)), \end{aligned}$$

as well as for dynamic problems,

$$\begin{aligned} F_i^x(x, t) &:= k_i^x r'_{x,i}(x, t), \\ F_i^y(x, t) &:= k_i^s (r'_{y,i}(x, t) - \varphi_{z,i}(x, t)), \\ F_i^z(x, t) &:= k_i^s (r'_{z,i}(x, t) + \varphi_{y,i}(x, t)), \end{aligned}$$

Table 2.1: Physical parameters of the Timoshenko beam model

Parameter	Physical significance	Parameter	Physical significance
κ_i	shear coefficient	$k_i^x := \bar{E}_i \bar{A}_i$	axial stiffness
$\bar{G}_i (\text{N/m}^2)$	shear modulus	$k_i^s := \kappa_i \bar{G}_i \bar{A}_i$	shear stiffness
$\bar{A}_i (\text{m}^2)$	cross-section area	$c_i^x := \bar{G}_i \bar{I}_i^t$	torsional stiffness
$\bar{E}_i (\text{N/m}^2)$	elastic modulus	$c_i^y := \bar{E}_i \bar{I}_i^y$	bending stiffnesses (w.r.t. the y -axis)
$\bar{I}_i^t (\text{m}^4)$	torsion constant	$c_i^z := \bar{E}_i \bar{I}_i^z$	bending stiffnesses (w.r.t. the z -axis)
$\bar{I}_i^y (\text{m}^4)$	second moments of area (w.r.t. the y -axis)	$\rho_i (\text{kg/m}^3)$	mass per unit length
$\bar{I}_i^z (\text{m}^4)$	second moments of area (w.r.t. the z -axis)		

and the internal moments $M_i^x, M_i^y, M_i^z : [0, L_i] \rightarrow \mathbb{R}^3$ by

$$\begin{aligned} M_i^x(x) &:= c_i^x \varphi'_{x,i}(x), \\ M_i^y(x) &:= c_i^y \varphi'_{y,i}(x), \\ M_i^z(x) &:= c_i^z \varphi'_{z,i}(x), \end{aligned}$$

as well as for dynamic problems

$$\begin{aligned} M_i^x(x, t) &:= c_i^x \varphi'_{x,i}(x, t), \\ M_i^y(x, t) &:= c_i^y \varphi'_{y,i}(x, t), \\ M_i^z(x, t) &:= c_i^z \varphi'_{z,i}(x, t). \end{aligned}$$

These functions can also be represented in global coordinates as:

$$\begin{aligned} F_i(x) &:= \mathbb{E}_i(\phi) (F_i^x(x), F_i^y(x), F_i^z(x))^\top, \\ M_i(x) &:= \mathbb{E}_i(\phi) (M_i^x(x), M_i^y(x), M_i^z(x))^\top. \end{aligned}$$

For dynamic, we have

$$\begin{aligned} F_i(x, t) &:= \mathbb{E}_i(\phi_t) (F_i^x(x, t), F_i^y(x, t), F_i^z(x, t))^\top, \\ M_i(x, t) &:= \mathbb{E}_i(\phi_t) (M_i^x(x, t), M_i^y(x, t), M_i^z(x, t))^\top. \end{aligned}$$

These equations show the intricate interplay between internal forces and moments, offering invaluable insights into the behavior of Timoshenko beam structures.

2.2.2 Dynamic Modeling of Timoshenko Beams

We use the notation $\partial_{tt}r := \ddot{r}$ and $\partial_{xx}r := r''$. The PDE dynamic equations of the Timoshenko beam model in 3D in local coordinates are described in the following forms([81]),

$$\rho_i \bar{A}_i \ddot{r}_{x,i}(x, t) = k_i^x r_{x,i}''(x, t) + f_{x,i}(x, t), \quad (2.9a)$$

$$\rho_i \bar{I}_i^t \ddot{\varphi}_{x,i}(x, t) = c_i^x \varphi_{x,i}''(x, t) + m_{x,i}(x, t), \quad (2.9b)$$

$$\rho_i \bar{A}_i \ddot{r}_{y,i}(x, t) = k_i^s (r_{y,i}''(x, t) - \varphi_{z,i}'(x, t)) + f_{y,i}(x, t), \quad (2.9c)$$

$$\rho_i \bar{A}_i \ddot{r}_{z,i}(x, t) = k_i^s (r_{z,i}''(x, t) + \varphi_{y,i}'(x, t)) + f_{z,i}(x, t), \quad (2.9d)$$

$$\rho_i \bar{I}_i^y \ddot{\varphi}_{y,i}(x, t) = c_i^y \varphi_{y,i}''(x, t) - k_i^s (r_{z,i}'(x, t) + \varphi_{y,i}(x, t)) + m_{y,i}(x, t) \quad (2.9e)$$

$$\rho_i \bar{I}_i^z \ddot{\varphi}_{z,i}(x, t) = c_i^z \varphi_{z,i}''(x, t) + k_i^s (r_{y,i}'(x, t) - \varphi_{z,i}(x, t)) + m_{z,i}(x, t). \quad (2.9f)$$

We also call this a *full model*. This system comprises six equations that govern the displacements and rotations of the beam. The first two equations relate to the displacements along the x -axis and rotations about the x -axis, respectively. The next two equations are concerned with the forces acting along the y and z axes, respectively. And the final two relate to the moments.

We employ the simplified diagonal uncoupled constitutive laws if there is no initial shear and either the initial twist is zero or the second moments of area coincide

$$C_N := \begin{pmatrix} k_i^x & 0 & 0 \\ 0 & k_i^s & 0 \\ 0 & 0 & k_i^s \end{pmatrix}, \quad C_M := \begin{pmatrix} c_i^x & 0 & 0 \\ 0 & c_i^y & 0 \\ 0 & 0 & c_i^z \end{pmatrix}.$$

The material inertia matrix is given by

$$C_I = \begin{pmatrix} \bar{I}_i^t & 0 & 0 \\ 0 & \bar{I}_i^y & 0 \\ 0 & 0 & \bar{I}_i^z \end{pmatrix}.$$

The Eq. (2.9) in global coordinates can be written as

$$\begin{aligned} \rho_i \bar{A}_i \ddot{r}_i(x, t) &= C_N (r_i''(x, t) + \hat{e}_{i_1} \varphi_i'(x, t)) + F_i(x, t), \\ \rho_i C_I \ddot{\varphi}_i &= C_M \varphi_i(x, t) + \hat{e}_{i_1} C_N (r_i'(x, t) + \hat{e}_{i_1} \varphi_i(x, t)) + M_i(x, t). \end{aligned} \quad (2.10)$$

Beam models are employed to characterize thin deformable structures, implying that, according to common understanding, their characteristic dimension is significantly greater in one spatial direction than in the perpendicular plane. Using the geometrical attributes of beams facilitates the establishment of a constrained motion, leading to the reduction of the full three-dimensional unknowns to spatially one-dimensional quantities based on specific assumptions. For example, we reduce the three-dimensional beam equations to one-dimension in the following case with governing PDE holding for $(x, t) \in [0, L_i] \times [0, T]$ (See [51]):

$$\begin{aligned} \rho_i \bar{A}_i \ddot{r}_{z,i}(x, t) &= k_i^s (r_{z,i}''(x, t) + \varphi_{y,i}'(x, t)) + f_{z,i}(x, t), \\ \rho_i \bar{I}_i^y \ddot{\varphi}_{y,i}(x, t) &= c_i^y \varphi_{y,i}''(x, t) - k_i^s (r_{z,i}'(x, t) + \varphi_{y,i}(x, t)) + m_{y,i}(x, t). \end{aligned} \quad (2.11)$$

There is no transformation from local coordinate to global. We only consider the $r_{z,i}$ and $\varphi_{y,i}$. The Timoshenko beam model of concern is shown in Fig. 2.2 and we call this the *reduced model*. At the origin 0, the deflection and the rotational angle are zero for time t , namely, $r_{z,i}(0, t) = \varphi_{y,i}(0, t) = 0$.

Assumption 1 The axial deformation of the Timoshenko beam is not taken into account in this model.

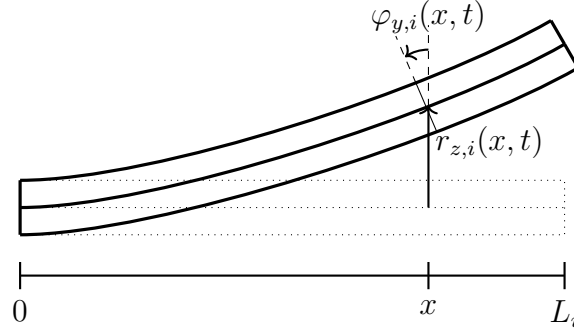


Figure 2.2: Displacement and bending of a single beam.

2.2.3 Static Modeling of Timoshenko Beams

In addition to the dynamic model, we extend our analysis to the static model, providing a comprehensive perspective on the beam's behavior under varying conditions. In 3D, the static Timoshenko beam equation system for a single beam in local coordinate is represented in Eq. (2.12),

$$f_{x,i}(x) + k_i^x r_{x,i}''(x) = 0, \quad (2.12a)$$

$$m_{x,i}(x) + c_i^x \varphi_{x,i}''(x) = 0, \quad (2.12b)$$

$$f_{y,i}(x) + k_i^s (r_{y,i}''(x) - \varphi_{z,i}'(x)) = 0, \quad (2.12c)$$

$$f_{z,i}(x) + k_i^s (r_{z,i}''(x) + \varphi_{y,i}'(x)) = 0, \quad (2.12d)$$

$$m_{y,i}(x) + c_i^y \varphi_{y,i}''(x) - k_i^s (r_{z,i}'(x) + \varphi_{y,i}(x)) = 0, \quad (2.12e)$$

$$m_{z,i}(x) + c_i^z \varphi_{z,i}''(x) + k_i^s (r_{y,i}'(x) - \varphi_{z,i}(x)) = 0. \quad (2.12f)$$

Similarly, the static system in the global coordinate is given by,

$$\begin{aligned} C_N(r_i''(x, t) + \hat{e}_{i1}\varphi_i'(x, t)) + F_i(x, t) &= 0, \\ C_M\varphi_i(x, t) + \hat{e}_{i1}C_N(r_i'(x, t) + \hat{e}_{i1}\varphi_i(x, t)) + M_i(x, t) &= 0. \end{aligned} \quad (2.13)$$

The reduced system is described in the following forms:

$$\begin{aligned} f_{z,i}(x) + k_i^s (r_{z,i}''(x) + \varphi_{y,i}'(x)) &= 0, \\ m_{y,i}(x) + c_i^y \varphi_{y,i}''(x) - k_i^s (r_{z,i}'(x) + \varphi_{y,i}(x)) &= 0. \end{aligned} \quad (2.14)$$

This static model provides additional insights into the structural response of Timoshenko beams and complements the dynamic analysis by considering the beam's behavior under equilibrium conditions. Our optimal control problem is formulated based on this static model, thus highlighting its significance in guiding optimization strategies for Timoshenko beam systems.

Chapter 3

Necessary and Sufficient Optimality Conditions for the Nonlinear Model

Networks are governed by state equations on metric graphs. Evaluation state equations are of type wave equation, i.e., hyperbolic partial differential equations static state equations are ordinary differential equations. We consider optimal control problems for evolution state equations. In such a case, the so-called Turnpike Poverty occurs, therefore the static state equation can be used for control and design of the network.

In this chapter, the optimization problems are considered for nonlinear steady state ordinary differential equations. The results obtained can be applied to control and design of networks with nonlinear evolution equations. The results are obtained in the framework of local Pontryagin maximum principle, see [63].

The control and shape problems are considered within the same mathematical framework for necessary and sufficient optimality conditions. For numerical methods, two approaches are possible. The first is approximate then optimize. The second is optimized and then approximate. Numerical results are obtained for the latter approach.

The results are published in [63].

3.1 Preliminaries

3.1.1 Formulation of the First-Order Necessary Optimality Conditions for an Autonomous Problem on the Interval $[0, 1]$

Consider the following autonomous problem of optimal control:

$$\left. \begin{aligned} J(x, u) &= \int_0^1 F(x(t), u(t))dt \rightarrow \min, \\ \dot{x}(t) &= f(x(t), u(t)) \quad \forall t \in [0, 1], \quad \kappa(x(0), x(1)) \leq 0, \quad K(x(0), x(1)) = 0. \end{aligned} \right\} \quad (3.1)$$

Here $x : [0, 1] \rightarrow \mathbb{R}^n$ is a continuously differentiable function, $u : [0, 1] \rightarrow \mathbb{R}^m$ is a continuous function, and $\dot{x} = dx/dt$. Hence the problem is considered in the space

$$\mathcal{W} := C^1([0, 1], \mathbb{R}^n) \times C([0, 1], \mathbb{R}^m).$$

A local minimum in this space is called a *weak local minimum*. We call x the *state variable* and u the *control*. All data $F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}$, $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$, $\kappa : \mathbb{R}^{2n} \rightarrow \mathbb{R}^k$, $K : \mathbb{R}^{2n} \rightarrow \mathbb{R}^s$ are assumed to be continuously differentiable.

We say that $w = (x, u) \in \mathcal{W}$ is an *admissible point* if it satisfies all the constraints of the problem. For brevity we set $\xi = (x(0), x(1))$.

Let us formulate first-order necessary optimality conditions for this problem. We introduce the Hamiltonian (Pontryagin) function and the endpoint Lagrange function:

$$H(x, u, p, \alpha_0) = pf(x, u) + \alpha_0 F(x, u), \quad L = \alpha\kappa(\xi) + \beta K(\xi),$$

where p, α, β are row vectors of the same dimensions as the column vectors f, κ, K , respectively, α_0 is a number. By definition, $pf = \sum_{i=1}^n p_i f_i$, where p_i and f_i are the components of the vectors p and f , respectively.

Denote by $\mathbb{R}^{n\top}$ the space of row vectors of dimension n .

By F_x and F_u we denote the partial derivatives $\partial F/\partial x$ and $\partial F/\partial u$, respectively, considered as row vectors, i.e. $F_x \in \mathbb{R}^{n\top}$, $F_u \in \mathbb{R}^{m\top}$. Similarly, $f_x := \partial f/\partial x$ and $f_u := \partial f/\partial u$, which are matrices of order $n \times n$ and $n \times m$, respectively. Note that $H_x \in \mathbb{R}^{n\top}$, $H_u \in \mathbb{R}^{m\top}$ are row vectors, and $H_p = f \in \mathbb{R}^n$ is a column vector.

We say that at an admissible point $w^0 = (x^0, u^0) \in \mathcal{W}$ the *local minimum principle* (LMP) is satisfied if there exists a continuously differentiable function $p : [0, 1] \rightarrow \mathbb{R}^{n\top}$, a number α_0 , and row vectors $\alpha \in \mathbb{R}^{k\top}$, $\beta \in \mathbb{R}^{s\top}$ such that the following system of optimality conditions holds:

- (a) the nonnegativity conditions: $\alpha_0 \geq 0, \alpha \geq 0$,
- (b) the nontriviality condition: $\alpha_0 + |\alpha| + |\beta| > 0$,
- (c) the complementary slackness condition: $\alpha\kappa(\xi^0) = 0$, where $\xi^0 = (x^0(0), x^0(1))$,
- (d) the adjoint equation: $-\dot{p}(t) = H_x(w^0(t), p(t), \alpha_0) \forall t \in [0, 1]$,
- (e) the transversality conditions: $(-p(0), p(1)) = L_\xi(\xi^0, \alpha, \beta)$,
- (f) the stationarity of the Hamiltonian with respect to the control:
 $H_u(w^0(t), p(t), \alpha_0) = 0 \forall t \in [0, 1]$.

From the equation $\dot{x}^0 = f(w^0)$ and conditions (d) and (f) it follows

- (g) the *condition for the Hamiltonian to be constant*: there exists a constant c_H such that $H(w^0(t), p(t), \alpha_0) = c_H \forall t \in [0, 1]$.

Indeed, $\frac{d}{dt}H(w^0(t), p(t), \alpha_0) = H_x(w^0(t), p(t), \alpha_0)\dot{x}^0(t) + H_u(w^0(t), p(t), \alpha_0)\dot{u}^0(t) + \dot{p}(t)H_p(w^0(t), p(t), \alpha_0) = -\dot{p}(t)\dot{x}^0(t) + \dot{p}(t)\dot{x}^0(t) = 0$. $\square$

The following theorem is well known, see, for example, [1], [16], [54], [53], [66].

Theorem 1. *If w^0 is a weak local minimum in problem (3.1), then it satisfies the LMP.*

The case, when the cost Lagrange multiplier α_0 is not equal to zero (for any quadruple $(\alpha_0, \alpha, \beta, p(\cdot))$ satisfying the LMP conditions), is called *normal*. Let us formulate a condition that guarantees the normal case for the point w^0 . Introduce a *set of active indices*

$$I = \{i \in \{1, \dots, k\} : \kappa_i(\xi^0) = 0\}.$$

We say that the *Mangasyan-Fromowitz constraint qualification* (MFCQ) is satisfied for the point $w^0 = (x^0, u^0) \in \mathcal{W}$ if there exists a pair $(x, u) \in \mathcal{W}$ such that

$$\kappa'_i(\xi^0)\xi < 0 \quad \forall i \in I, \quad K'(\xi^0)\xi = 0, \quad \xi = (x(0), x(1)), \quad \dot{x} = f'(w^0)w,$$

where, for example, $f'(w^0)w = f_x(w^0)x + f_u(w^0)u$. In this case, in the LMP conditions, we can set $\alpha_0 = 1$.

3.1.2 Formulation of the Second-Order Sufficient Optimality Conditions for an Autonomous Problem on the Interval $[0, 1]$

Consider again the autonomous problem (3.1). Now we suppose that all data F, f, κ, K are twice continuously differentiable.

Let us formulate sufficient second-order conditions for a weak local minimum at an admissible point $w^0 = (x^0, u^0) \in \mathcal{W}$, satisfying necessary first-order conditions with the adjoint variable p and Lagrange multipliers α_0, α, β . Define the *critical cone* at the point w^0 :

$$\mathcal{C} := \{\delta w = (\delta x, \delta u) \in \mathcal{W} : \quad \delta \dot{x}(t) = f'(w^0(t))\delta w(t), \quad K'(\xi^0)\delta \xi = 0, \\ \kappa'_i(\xi^0)\delta \xi \leq 0, \quad i \in I, \quad \int_0^1 F'(w^0(t))\delta w(t)dt \leq 0\},$$

where $\delta \xi = (\delta x(0), \delta x(1))$. The equation $\delta \dot{x} = f'(w^0)\delta w$ is called the *equation in variations*.

In the normal case, where $\alpha_0 = 1$, the inequality $\int_0^1 F'(w^0(t))\delta w(t)dt \leq 0$ can be excluded from the definition of the critical cone, but then we must add the equalities $\alpha_i \kappa'_i(\xi^0)\delta \xi = 0, i \in I$. Thus, in the normal case, we have

$$\mathcal{C} := \{\delta w = (\delta x, \delta u) \in \mathcal{W} : \quad \delta \dot{x}(t) = f'(w^0(t))\delta w(t), \quad K'(\xi^0)\delta \xi = 0, \\ \kappa'_i(\xi^0)\delta \xi \leq 0, \quad i \in I, \quad \alpha_i \kappa'_i(\xi^0)\delta \xi = 0, \quad i \in I\}.$$

This is easy to prove using the LMP conditions. Later, in Section 3.2, where we consider the normal case, we will use this critical cone representation.

Define the *strengthened Legendre condition*: there exists $c_L > 0$ such that for all $t \in [0, 1]$ we have $\langle H_{uu}(w^0(t), p(t), \alpha_0)u, u \rangle \geq c_L |u|^2 \quad \forall u \in \mathbb{R}^m$. Here $H_{uu} = \partial^2 H / \partial u^2$ stands for the second partial derivative of H with respect to the control.

Next, define a *quadratic form*:

$$2\Omega(\delta w) = \langle L_{\xi\xi}(\xi^0, \alpha, \beta)\delta\xi, \delta\xi \rangle + \int_0^1 \langle H_{ww}(w^0(t), p(t), \alpha_0)\delta w(t), \delta w(t) \rangle dt.$$

Note that if $\kappa(\xi)$ and $K(\xi)$ are affine functions, then $L = \alpha\kappa + \beta K$ is also an affine function of ξ , and therefore, $L_{\xi\xi} = 0$. In this case, the endpoint term $\langle L_{\xi\xi}(\xi^0, \alpha, \beta)\delta\xi, \delta\xi \rangle$ vanishes, and Ω reduces to the integral only.

The following theorem holds, see, for example, [62].

Theorem 2. *Assume that for the point w^0*

(a) *the strengthened Legendre condition is satisfied,*

(b) *there exists a constant $c_\Omega > 0$ such that $\Omega(\delta w) \geq c_\Omega(|\delta x(0)|^2 + \|\delta u\|_2^2) \quad \forall \delta w \in \mathcal{C}$.*

Then there are $c > 0$ and $\varepsilon > 0$ such that $J(w) - J(w^0) \geq c(\|x - x^0\|_\infty^2 + \int_0^1 |u(t) - u^0(t)|^2 dt)$ for all admissible $w = (x, u)$ such that $\|w - w^0\|_\infty < \varepsilon$, and hence w^0 is a weak local minimum in the problem.

Remark 1. *Since $\Omega(-\delta w) = \Omega(\delta w)$ for all $\delta w \in \mathcal{W}$, condition (b) in this theorem is equivalent to the condition $\Omega(\delta w) \geq c_\Omega(|\delta x(0)|^2 + \|\delta u\|_2^2) \quad \forall \delta w \in \Sigma$, where $\Sigma = \mathcal{C} \cup (-\mathcal{C})$. In particular, let $\mathcal{C} = \{\delta w \in \Gamma, l(\delta w) \leq 0\}$, where Γ is a subspace, and l is a linear functional. Then, obviously, $\Sigma = \Gamma$.*

3.1.3 Matrix Riccati Equation

Now we consider a sufficient condition for positive definiteness of the quadratic form Ω on the subspace Γ . Assume that Γ has the form:

$$\Gamma = \left\{ \delta w = (\delta x, \delta u) \in \mathcal{W} : \delta \dot{x} = f_x(w^0)\delta x + f_u(w^0)\delta u, \mathcal{E}\delta\xi = 0 \right\},$$

where $\mathcal{E}$ is a constant matrix, $\delta\xi = (\delta x(0), \delta x(1))$. Let us show that the quadratic form Ω could be transformed into a perfect square if the corresponding Riccati equation has a solution $Q(t)$ defined on $[0, 1]$. Assume that the strengthened Legendre condition is satisfied. Define the Riccati matrix equation along $(x^0(t), u^0(t), p(t))$ by

$$\dot{Q} + Qf_x + f_x^T Q + H_{xx} - (H_{xu} + Qf_u)H_{uu}^{-1}(H_{ux} + f_u^T Q) = 0, \quad t \in [0, 1], \quad (3.2)$$

where $Q = Q(t)$ is a symmetric matrix of order n whose elements belong to C^1 , $f_x = f_x(w^0)$, $H_{xx} = H_{xx}(w^0, p, \alpha_0)$, etc., f_x^T means the transposed matrix f_x .

Theorem 3. *Assume that the strengthened Legendre condition is satisfied and there exists a symmetric solution Q (with the entries belonging to C^1) of the matrix Riccati equation*

on $[0, 1]$. Then the quadratic form Ω has the following transformation into a perfect square on the subspace Γ :

$$2\Omega(\delta w) = \int_0^1 \langle H_{uu}^{-1} \delta v, \delta v \rangle dt + \langle M \delta \xi, \delta \xi \rangle \quad \forall \delta w \in \Gamma, \quad (3.3)$$

where $\delta v := (H_{ux} + f_u^T Q) \delta x + H_{uu} \delta u$, H_{uu}^{-1} is the inverse matrix of matrix H_{uu} , and

$$M := \begin{pmatrix} L_{x_0 x_0} + Q(0) & L_{x_0 x_1} \\ L_{x_1 x_0} & L_{x_1 x_1} - Q(1) \end{pmatrix}.$$

For the reader's convenience, we give a proof of this theorem. We follow [62] (see also [52]).

Proof. Let $(\delta x, \delta u) \in \Gamma$. Then

$$\begin{aligned} 2\langle Q \delta \dot{x}, \delta x \rangle &= 2\langle Q(f_x \delta x + f_u \delta u), \delta x \rangle \\ &= \langle (Q f_x + f_x^T Q) \delta x, \delta x \rangle + \langle Q f_u \delta u, \delta x \rangle + \langle f_u^T Q \delta x, \delta u \rangle. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{d}{dt} \langle Q \delta x, \delta x \rangle &= \langle \dot{Q} \delta x, \delta x \rangle + 2\langle Q \delta \dot{x}, \delta x \rangle \\ &= \langle \dot{Q} \delta x, \delta x \rangle + \langle (Q f_x + f_x^T Q) \delta x, \delta x \rangle + \langle Q f_u \delta u, \delta x \rangle + \langle f_u^T Q \delta x, \delta u \rangle \\ &= \langle (\dot{Q} + Q f_x + f_x^T Q) \delta x, \delta x \rangle + \langle Q f_u \delta u, \delta x \rangle + \langle f_u^T Q \delta x, \delta u \rangle. \end{aligned}$$

Integrating over $[0, 1]$, we get

$$\begin{aligned} &\langle Q(1) \delta x(1), \delta x(1) \rangle - \langle Q(0) \delta x(0), \delta x(0) \rangle \\ &= \int_0^1 \left(\langle (\dot{Q} + Q f_x + f_x^T Q) \delta x, \delta x \rangle + \langle Q f_u \delta u, \delta x \rangle + \langle f_u^T Q \delta x, \delta u \rangle \right) dt. \end{aligned}$$

Consequently,

$$\begin{aligned} &\int_0^1 \left(\langle (\dot{Q} + Q f_x + f_x^T Q) \delta x, \delta x \rangle + \langle Q f_u \delta u, \delta x \rangle + \langle f_u^T Q \delta x, \delta u \rangle \right) dt \\ &+ \langle Q(0) \delta x(0), \delta x(0) \rangle - \langle Q(1) \delta x(1), \delta x(1) \rangle = 0. \end{aligned}$$

Adding this zero form to the form $2\Omega(\delta w)$, we obtain

$$\begin{aligned} 2\Omega(\delta w) &= \int_0^1 \left(\langle (\dot{Q} + Q f_x + f_x^T Q + H_{xx}) \delta x, \delta x \rangle \right. \\ &\quad \left. + \langle (Q f_u + H_{xu}) \delta u, \delta x \rangle + \langle (f_u^T Q + H_{ux}) \delta x, \delta u \rangle + \langle H_{uu} \delta u, \delta u \rangle \right) dt \\ &\quad + \langle Q(0) \delta x(0), \delta x(0) \rangle - \langle Q(1) \delta x(1), \delta x(1) \rangle + \langle L_{\xi\xi} \delta \xi, \delta \xi \rangle. \end{aligned}$$

Now let Q satisfy the Riccati equation (3.2). Then

$$\begin{aligned} 2\Omega(\delta w) &= \int_0^1 \left(\langle (H_{xu} + Q f_u) H_{uu}^{-1} (H_{ux} + f_u^T Q) \delta x, \delta x \rangle \right. \\ &\quad \left. + \langle (Q f_u + H_{xu}) \delta u, \delta x \rangle + \langle (f_u^T Q + H_{ux}) \delta x, \delta u \rangle + \langle H_{uu} \delta u, \delta u \rangle \right) dt \\ &\quad + \langle Q(0) \delta x(0), \delta x(0) \rangle - \langle Q(1) \delta x(1), \delta x(1) \rangle + \langle L_{\xi\xi} \delta \xi, \delta \xi \rangle. \end{aligned}$$

Since $\langle H_{uu}\delta u, \delta u \rangle = \langle (H_{uu})^{-1}H_{uu}\delta u, H_{uu}\delta u \rangle$ and $\langle Q(0)\delta x(0), \delta x(0) \rangle - \langle Q(1)\delta x(1), \delta x(1) \rangle + \langle L_{\xi\xi}\delta\xi, \delta\xi \rangle = \langle M\delta\xi, \delta\xi \rangle$, we obtain

$$\begin{aligned} 2\Omega(\delta w) &= \int_0^1 \left(\langle (H_{xu} + Qf_u)H_{uu}^{-1}(H_{ux} + f_u^\top Q)\delta x, \delta x \rangle \right. \\ &\quad + \langle (Qf_u + H_{xu})\delta u, \delta x \rangle + \langle (f_u^\top Q + H_{ux}\delta x, \delta u) \rangle + \langle (H_{uu})^{-1}H_{uu}\delta u, H_{uu}\delta u \rangle \Big) dt \\ &\quad + \langle M\delta\xi, \delta\xi \rangle. \end{aligned}$$

Further,

$$\begin{aligned} &\langle (H_{xu} + Qf_u)H_{uu}^{-1}(H_{ux} + f_u^\top Q)\delta x, \delta x \rangle + \langle (Qf_u + H_{xu})\delta u, \delta x \rangle \\ &\quad + \langle (f_u^\top Q + H_{ux}\delta x, \delta u) \rangle + \langle (H_{uu})^{-1}H_{uu}\delta u, H_{uu}\delta u \rangle \\ &= \langle (H_{uu})^{-1}((H_{ux} + f_u^\top Q)\delta x + H_{uu}\delta u), ((H_{ux} + f_u^\top Q)\delta x + H_{uu}\delta u) \rangle \\ &= \langle (H_{uu})^{-1}\delta v, \delta v \rangle, \end{aligned}$$

where $\delta v = (H_{ux} + f_u^\top Q)\delta x + H_{uu}\delta u$. Consequently, $2\Omega(\delta w) = \int_0^1 \langle (H_{uu})^{-1}\delta v, \delta v \rangle dt + \langle M\delta\xi, \delta\xi \rangle$. $\square$

Assume that M is non-negative definite. Recall that H_{uu} is positive definite, and then $(H_{uu})^{-1}$ is positive definite too. Hence $\Omega(\delta w) \geq 0$, $\forall \delta w = (\delta x, \delta u) \in \Gamma$.

Suppose that $\Omega(\delta w) = 0$ for some $\delta w = (\delta x, \delta u) \in \Gamma$. Then, given (3.3) both non-negative terms $\int_0^1 \langle (H_{uu})^{-1}\delta v, \delta v \rangle dt$ and $\langle M\delta\xi, \delta\xi \rangle$ are equal zero. Condition

$$\int_0^1 \langle (H_{uu})^{-1}\delta v, \delta v \rangle dt = 0$$

implies $\delta v = 0$, i.e. $(H_{ux} + f_u^\top Q)\delta x + H_{uu}\delta u = 0$. Hence $\delta u = -(H_{uu})^{-1}(H_{ux} + f_u^\top Q)\delta x$. It follows that δx is a solution to the homogeneous differential equation $\delta\dot{x} = f_x(\hat{w})\delta x - f_u(\hat{w})(H_{uu})^{-1}(H_{ux} + f_u^\top Q)\delta x$. Let us now assume that the conditions $\mathcal{E}\delta\xi = 0$, $\langle M\delta\xi, \delta\xi \rangle = 0$ imply that $\delta x(0) = 0$ or $\delta x(1) = 0$. Then $\delta x = 0$ and hence $\delta u = 0$. Consequently, $\Omega(\delta w) > 0$ for all $\delta w \in \Gamma \setminus \{0\}$. Since Ω is a Legendre form, its positiveness on the subspace Γ implies positive definiteness on Γ . Thus we obtain the following result.

Theorem 4. *Assume that the strengthened Legendre condition is satisfied and there exists a symmetric solution Q (with the entries belonging to C^1) of the Riccati matrix equation on $[0, 1]$ such that*

- (a) *the matrix M is non-negative definite;*
- (b) *for all $\xi = (x_0, x_1) \in \mathbb{R}^{2n}$ the conditions $\mathcal{E}\xi = 0$, $\langle M\xi, \xi \rangle = 0$ imply that $x_0 = 0$ or $x_1 = 0$.*

Then the quadratic form Ω is positive definite on the subspace Γ .

Other designations

Let Γ has the form:

$$\Gamma = \left\{ \delta w = (\delta x, \delta u) : \delta \dot{x}(t) = A(t)\delta x(t) + B(t)\delta u(t), \mathcal{E}\delta\xi = 0 \right\},$$

and

$$2\Omega(\delta w) = \langle N\delta\xi, \delta\xi \rangle + \int_0^1 (\langle R(t)\delta x(t), \delta x(t) \rangle + 2\langle S(t)\delta u(t), \delta x(t) \rangle + \langle U(t)\delta u(t), \delta u(t) \rangle) dt, \quad (3.4)$$

where $\mathcal{E}$ and N are constant matrices, $A(t)$, $B(t)$, $R(t)$, $S(t)$, $U(t)$ are matrices with continuous entries. Assume that the matrices $R(t)$ and $U(t)$ are symmetric and, moreover, the matrix $U(t)$ is positive definite for all $t \in [0, 1]$, and the constant symmetric matrix N of the order $2n$ has the form

$$N = \begin{pmatrix} N_{00} & N_{01} \\ N_{10} & N_{11} \end{pmatrix},$$

where N_{00} , N_{01} , N_{10} , N_{11} are constant $n \times n$ matrices, N_{00} and N_{11} are symmetric, and $N_{10} = N_{01}^\top$. Previously, we had $A = f_x$, $B = f_u$, $R = H_{xx}$, $S = H_{xu}$, $U = H_{uu}$. We can prove similar results for the new quadratic form and subspace in the same way as before. Now the Riccati equation and the matrix M are:

$$\dot{Q} + QA + A^\top Q + R - (S + QB)U^{-1}(S^\top + B^\top Q) = 0, \quad (3.5)$$

$$M = \begin{pmatrix} N_{00} + Q(0) & N_{01} \\ N_{10} & N_{11} - Q(1) \end{pmatrix}. \quad (3.6)$$

3.2 Statement of Control Problem for Single Beam

Consider the following optimal control problem. Let $z(x)$ be a state variable, $f(x)$ be a control, where $x \in [0, l]$. Here $z = (z_1, \dots, z_n)^\top \in \mathbb{R}^n$, $f \in \mathbb{R}^1$, $l > 0$. We assume that $z(x)$ is a continuously differentiable function and $f(x)$ is a continuous function. The control system has the form

$$\frac{dz(x)}{dx} = \varphi(z(x)) + e_1 f(x), \quad x \in [0, l], \quad K(z(0), z(l)) = 0 \quad (3.7)$$

where $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a twice continuously differentiable function, $e_1 = (1, 0, \dots, 0)^\top \in \mathbb{R}^n$, and $K : \mathbb{R}^{2n} \rightarrow \mathbb{R}^s$ is an affine function of its arguments $\zeta_0 := z(0)$ and $\zeta_l := z(l)$. Set $\zeta = (\zeta_0, \zeta_l)$. The cost that needs to be minimized is:

$$J = \int_0^l F(x, z(x), f(x)) dx, \quad (3.8)$$

where $F(x, z, f)$ is a twice continuously differentiable function. In this problem, l is not fixed but satisfies the constraint

$$l \in [a, b], \quad \text{where } 0 < a < b. \quad (3.9)$$

An arbitrary admissible process in this problem is defined by the triple $(l, z(\cdot), f(\cdot))$, where $z : [0, l] \rightarrow \mathbb{R}^n$, $f : [0, l] \rightarrow \mathbb{R}$. We will consider a fixed admissible process

$$(l^0, z^0(\cdot), f^0(\cdot)), \quad (3.10)$$

where z^0 and f^0 are defined on $[0, l^0]$.

Let us represent this problem as a problem on the interval $[0, 1]$. To do this, we use the following change of the independent variable x . Let $t \in [0, 1]$ be a new independent variable. We set

$$\tilde{x}(t) = lt, \quad t \in [0, 1].$$

Then $\tilde{x} : [0, 1] \rightarrow [0, l]$. We treat $\tilde{x}(t)$ as a new state variable. We also treat $l = \tilde{l}(t)$ as another state variable, constant on $[0, 1]$. Hence

$$\frac{d\tilde{l}(t)}{dt} = 0, \quad \frac{d\tilde{x}(t)}{dt} = \tilde{l}(t), \quad t \in [0, 1], \quad \tilde{x}(0) = 0.$$

To any admissible process (l, z, f) in the original problem, we associate the process $(\tilde{l}, \tilde{x}, \tilde{z}, \tilde{f})$ in the new problem by the formulas

$$\tilde{l}(t) = l, \quad \tilde{x}(t) = lt, \quad \tilde{z}(t) = z(\tilde{x}(t)) = z(lt), \quad \tilde{f}(t) = f(\tilde{x}(t)) = f(lt) \quad \forall t \in [0, 1].$$

This is one-to-one correspondence. In what follows, we will continue to use the tilde for variables in the interval $[0, 1]$.

Thus, we obtain an autonomous problem with a new independent variable $t \in [0, 1]$:

$$\frac{d\tilde{l}(t)}{dt} = 0, \quad \frac{d\tilde{x}(t)}{dt} = \tilde{l}(t), \quad t \in [0, 1], \quad (3.11)$$

$$\frac{d\tilde{z}(t)}{dt} = \tilde{l}(t)(\varphi(\tilde{z}(t)) + e_1 \tilde{f}(t)), \quad t \in [0, 1], \quad (3.12)$$

$$\tilde{x}(0) = 0, \quad K(\tilde{z}(0), \tilde{z}(1)) = 0, \quad (3.13)$$

$$-\tilde{l}(0) + a \leq 0, \quad \tilde{l}(0) - b \leq 0, \quad (3.14)$$

$$J = \int_0^1 \tilde{l}(t) F(\tilde{x}(t), \tilde{z}(t), \tilde{f}(t)) dt \rightarrow \min. \quad (3.15)$$

We study the local minimum at the point

$$(\tilde{l}^0(\cdot), \tilde{x}^0, \tilde{z}^0(\cdot), \tilde{f}^0(\cdot)), \quad (3.16)$$

such that

$$\tilde{l}^0(t) = l^0, \quad \tilde{x}^0(t) = l^0 t, \quad \tilde{z}^0(t) = z^0(l^0 t), \quad \tilde{f}^0(t) = f^0(l^0 t), \quad t \in [0, 1].$$

This point corresponds to the process (3.10) in the original problem (3.7)-(3.9). Clearly, the minimum at (3.10) in problem (3.7)-(3.9) implies the minimum at (3.16) in problem (3.11)-(3.15) and vice versa.

Local Minimum Principle for Problem with One Beam

Denote by $\tilde{p}^z(t)$ the adjoint variable which corresponds to the equation for $\tilde{z}$ in the new problem. We consider $\tilde{p}^z = (\tilde{p}_1^z, \dots, \tilde{p}_n^z)$ as a row vector. We also introduce one-dimensional adjoint variables $\tilde{p}^x(t)$ and $\tilde{p}^l(t)$. The Hamiltonian and the endpoint Lagrange function are:

$$\begin{aligned}\tilde{H}(\tilde{l}, \tilde{x}, \tilde{z}, \tilde{f}, \tilde{p}^l, \tilde{p}^x, \tilde{p}^z, \alpha_0) &= \tilde{p}^x \tilde{l} + \tilde{p}^z \tilde{l}(\varphi(\tilde{z}) + e_1 \tilde{f}) + \alpha_0 \tilde{l} F(\tilde{x}, \tilde{z}, \tilde{f}), \\ \tilde{L} &= \alpha_a(-\tilde{l}(0) + a) + \alpha_b(\tilde{l}(0) - b) + \beta_x \tilde{x}(0) + \beta K(\tilde{z}(0), \tilde{z}(1)).\end{aligned}$$

Note that $\tilde{L}$ is an affine function of the endpoint values $\tilde{l}(0)$, $\tilde{x}(0)$, $\tilde{z}(0)$, $\tilde{l}(1)$, $\tilde{x}(1)$, $\tilde{z}(1)$ of the states $\tilde{l}$, $\tilde{x}$, and $\tilde{z}$, since K is an affine function by assumption.

Let us write down the first-order necessary optimality conditions at the point (3.16) in problem (3.11)-(3.15). The partial derivatives of $\tilde{H}$ with respect to $\tilde{l}$, $\tilde{x}$, $\tilde{z}$, $\tilde{f}$ have the form

$$\begin{aligned}\tilde{H}_{\tilde{l}} &= \tilde{p}^x + \tilde{p}^z(\varphi(\tilde{z}) + e_1 \tilde{f}) + \alpha_0 F(\tilde{x}, \tilde{z}, \tilde{f}), \\ \tilde{H}_{\tilde{x}} &= \alpha_0 \tilde{l} F_{\tilde{x}}(\tilde{x}, \tilde{z}, \tilde{f}), \\ \tilde{H}_{\tilde{z}} &= \tilde{p}^z \tilde{l} \varphi'(\tilde{z})^T + \alpha_0 \tilde{l} F_{\tilde{z}}(\tilde{x}, \tilde{z}, \tilde{f}), \\ \tilde{H}_{\tilde{f}} &= \tilde{p}^z \tilde{l} e_1 + \alpha_0 \tilde{l} F_{\tilde{f}}(\tilde{x}, \tilde{z}, \tilde{f}).\end{aligned}$$

- (a) The nonnegativity conditions: $\alpha_0 \geq 0$, $\alpha_a \geq 0$, $\alpha_b \geq 0$.
- (b) The nontriviality condition: $\alpha_0 + \alpha_a + \alpha_b + |\beta_x| + |\beta| > 0$.
- (c) The complementary slackness conditions: $\alpha_a(\tilde{l}^0(0) - a) = 0$, $\alpha_b(\tilde{l}^0(0) - b) = 0$.
- (d) The adjoint equations:

$$-\frac{d\tilde{p}^l(t)}{dt} = \tilde{p}^x(t) + \tilde{p}^z(t)(\varphi(\tilde{z}^0(t)) + e_1 \tilde{f}^0(t)) + \alpha_0 F(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)), \quad (3.17)$$

$$-\frac{d\tilde{p}^x(t)}{dt} = \alpha_0 \tilde{l}^0 F_{\tilde{x}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)), \quad (3.18)$$

$$-\frac{d\tilde{p}^z(t)}{dt} = \tilde{p}^z(t) \tilde{l}^0 \varphi'(\tilde{z}^0(t)) + \alpha_0 \tilde{l}^0 F_{\tilde{z}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)), \quad t \in [0, 1]. \quad (3.19)$$

- (e) The transversality conditions:

$$\begin{aligned}-\tilde{p}^l(0) &= -\alpha_a + \alpha_b, & \tilde{p}^l(1) &= 0, \\ -\tilde{p}^x(0) &= \beta_x, & \tilde{p}^x(1) &= 0, \\ -\tilde{p}^z(0) &= \beta K_{\tilde{\zeta}_0}(\tilde{z}^0(0), \tilde{z}^0(1)), & \tilde{p}^z(1) &= \beta K_{\tilde{\zeta}_1}(\tilde{z}^0(0), \tilde{z}^0(1)),\end{aligned}$$

where $\tilde{\zeta}_0 = \tilde{z}(0)$, $\tilde{\zeta}_1 = \tilde{z}(1)$.

- (f) The condition $\tilde{H}_{\tilde{f}} = 0$: $\tilde{p}^z(t)\tilde{l}^0 e_1 + \alpha_0 \tilde{l}^0 F_{\tilde{f}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = 0$. Since $\tilde{l}^0 > 0$ and $\tilde{p}^z(t)e_1 = \tilde{p}_1^z(t)$, we get

$$\tilde{p}_1^z(t) + \alpha_0 \tilde{l}^0 F_{\tilde{f}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = 0, \quad t \in [0, 1].$$

- (g) Finally, the condition $\tilde{H} = \text{const}$ has the form: there exists a constant $\tilde{c}_H$ such that

$$\tilde{p}^x(t)\tilde{l}^0 + \tilde{p}^z(t)\tilde{l}^0 (\varphi(\tilde{z}^0(t)) + e_1 \tilde{f}^0(t)) + \alpha_0 \tilde{l}^0 F(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = \tilde{c}_H \quad \forall t \in [0, 1].$$

Denote the left hand side of this equality by $\tilde{H}(t)$. Dividing this equality by $\tilde{l}^0$, we obtain

$$\tilde{p}^x(t) + \tilde{p}^z(t)(\varphi(\tilde{z}^0(t)) + e_1 \tilde{f}^0(t)) + \alpha_0 F(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = \frac{\tilde{c}_H}{\tilde{l}^0} \quad \forall t \in [0, 1].$$

Integrating equation (3.17) over the interval $[0, 1]$ and using the above condition, we get $\tilde{p}^l(0) - \tilde{p}^l(1) = \frac{\tilde{c}_H}{\tilde{l}^0}$. This and the transversality conditions $-\tilde{p}^l(0) = -\alpha_a + \alpha_b$, $\tilde{p}^l(1) = 0$ give

$$\frac{\tilde{c}_H}{\tilde{l}^0} = \alpha_a - \alpha_b.$$

This relation means the following.

- (1) If $a < \tilde{l}^0 < b$ then by the complementary slackness conditions (c) we have $\alpha_a = \alpha_b = 0$ and therefore $\tilde{c}_H = 0$.
- (2) If $\tilde{l}^0 = a$, then by (c) we have $\alpha_b = 0$ and, therefore, $\tilde{c}_H = \alpha_a \tilde{l}^0 \geq 0$.
- (3) If $\tilde{l}^0 = b$, then by (c) we have $\alpha_a = 0$ and, therefore, $\tilde{c}_H = -\alpha_b \tilde{l}^0 \leq 0$.
- (4) Moreover, if $\tilde{c}_H > 0$, then $\alpha_a > 0$, and, therefore, by (c) $\tilde{l}^0 = a$; if $\tilde{c}_H < 0$, then $\alpha_b > 0$, and, therefore, by (c) $\tilde{l}^0 = b$.

Note that the transversality condition $\tilde{p}^x(1) = 0$ and adjoint equation (3.18) imply

$$\tilde{p}^x(t) = \alpha_0 \tilde{l}^0 \int_t^1 F_{\tilde{x}}(\tilde{z}^0(\tau), \tilde{f}^0(\tau)) d\tau, \quad t \in [0, 1]. \quad (3.20)$$

Thus, we obtain the following result. If (3.16) is a local minimum in problem (3.11)–(3.15), then there exist a number $\alpha_0 \geq 0$, a row vector $\beta \in \mathbb{R}^{s^\top}$, and a continuously such that the following system of optimality conditions holds:

$$\begin{aligned} \frac{d\tilde{z}}{dt} &= \tilde{l}^0 (\varphi(\tilde{z}^0(t)) + e_1 \tilde{f}^0(t)), \quad t \in [0, 1], \quad K(\tilde{z}^0(0), \tilde{z}^0(1)) = 0, \\ -\frac{d\tilde{p}^z}{dt} &= \tilde{p}^z(t)\tilde{l}^0 \varphi'(\tilde{z}^0(t)) + \alpha_0 \tilde{l}^0 F_{\tilde{z}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)), \quad t \in [0, 1], \\ -\tilde{p}^z(0) &= \beta K_{\tilde{z}_0}(\tilde{z}^0(0), \tilde{z}^0(1)), \quad \tilde{p}^z(1) = \beta K_{\tilde{z}_1}(\tilde{z}^0(0), \tilde{z}^0(1)), \\ \tilde{p}_1^z(t) &+ \alpha_0 \tilde{l}^0 F_{\tilde{f}}(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = 0, \quad t \in [0, 1]. \end{aligned}$$

These conditions imply the condition of the constancy of the Hamiltonian: there exists a constant $\tilde{c}_H$ such that

$$\tilde{p}^x(t)\tilde{l}^0 + \tilde{p}^z(t)\tilde{l}^0(\varphi(\tilde{z}^0(t)) + e_1\tilde{f}^0(t)) + \alpha_0\tilde{l}^0 F(\tilde{x}^0(t), \tilde{z}^0(t), \tilde{f}^0(t)) = \tilde{c}_H \quad \forall t \in [0, 1],$$

where $\tilde{p}^x(t)$ is defined by (3.20).

Moreover, the following is true. If $a < \tilde{l}^0 < b$, then $\tilde{c}_H = 0$. If $\tilde{l}^0 = a$, then $\tilde{c}_H \geq 0$; if $\tilde{c}_H > 0$, then $\tilde{l}^0 = a$. If $\tilde{l}^0 = b$, then $\tilde{c}_H \leq 0$; if $\tilde{c}_H < 0$, then $\tilde{l}^0 = b$. We now equivalently represent this system on the interval $[0, l^0]$. Introduce a function $p^z : [0, l^0] \rightarrow \mathbb{R}^{n^\top}$ such that $\tilde{p}^z(t) = p^z(\tilde{x}^0(t)) = p^z(l^0 t)$, that is $p^z(x) = \tilde{p}^z\left(\frac{x}{l^0}\right)$, $x \in [0, l^0]$. Then $\frac{d\tilde{p}^z}{dt} = \frac{dp^z}{dx}l^0$. Hence the adjoint equation for $\tilde{p}^z$ takes the form

$$-\frac{dp^z(x)}{dx} = p^z(x)\varphi'(z^0(x)) + \alpha_0 F_z(x^0(t), z^0(t), f^0(t)), \quad x \in [0, l^0].$$

So, the obtained result has the following formulation on the interval $[0, l^0]$. Below we replace p^z with p , and we also replace $(l^0, z^0(\cdot), f^0(\cdot))$ with $(l, z(\cdot), f(\cdot))$ omitting the superscript zero.

Theorem 5. *If $(l, z(\cdot), f(\cdot))$ is a local minimum in problem (3.7)-(3.9), then there exist a number $\alpha_0 \geq 0$, a row vector $\beta \in \mathbb{R}^{s^\top}$, and a continuously differentiable function $p : [0, l] \rightarrow \mathbb{R}^{n^\top}$ such that the following system of optimality conditions holds:*

$$\begin{aligned} \frac{dz(x)}{dx} &= \varphi(z(x)) + e_1 f(x), \quad x \in [0, l], \quad l \in [a, b], \quad K(z(0), z(l)) = 0, \\ -\frac{dp(x)}{dx} &= p(x)\varphi'(z(x)) + \alpha_0 F_z(x, z(x), f(x)), \quad x \in [0, l], \\ -p(0) &= \beta K_{\zeta_0}(z(0), z(1)), \\ p(l) &= \beta K_{\zeta_1}(z(0), z(l)), \\ p_1(x) + \alpha_0 F_f(x, z(x), f(x)) &= 0, \quad x \in [0, l]. \end{aligned}$$

These conditions imply the condition of the constancy of the Hamiltonian: there exists a constant c_H such that

$$p^x(x) + p(x)(\varphi(z(x)) + e_1 f(x)) + \alpha_0 F(x, z(x), f(x)) = c_H \quad \forall x \in [0, l],$$

where $p^x(x) = \alpha_0 \int_x^l F_x(y, z(y), f(y))dy$, $x \in [0, l]$. Moreover, the following is true. If $a < l < b$, then $c_H = 0$. If $l = a$, then $c_H \geq 0$; if $c_H > 0$, then $l = a$. If $l = b$, then $c_H \leq 0$; if $c_H < 0$, then $l = b$.

Since $p^x(l) = 0$ and $c_H = H(l)$, we get

$$c_H = p(l)(\varphi(z(l)) + e_1 f(l)) + \alpha_0 F(l, z(l), f(l)).$$

This formula does not use the adjoint variable p^x .

In what follows, we consider the case of

$$F(x, z, f) = \frac{1}{2}|z - z^*(x)|^2 + \frac{1}{2}(f - f^*(x))^2, \quad (3.21)$$

where $|z| = \sqrt{\langle z, z \rangle}$ and $z^*(x)$ and $f^*(x)$ are twice continuously differentiable functions defined on $[0, b]$.

Second-Order Sufficient Conditions for Problem with One Beam

For problem (3.11) -(3.15) on $[0, 1]$ with the function F defined by formula (3.21), we formulate sufficient second-order conditions for a weak local minimum at the point $\tilde{w}(\cdot) = (\tilde{l}(\cdot), \tilde{x}(\cdot), \tilde{z}(\cdot), \tilde{f}(\cdot))$.

Now suppose that the normal case holds for this point. Therefore, there are a row vector $\beta \in \mathbb{R}^{s^\top}$ and a continuously differentiable function $\tilde{p} : [0, 1] \rightarrow \mathbb{R}^{n^\top}$ such that the necessary optimality conditions in Section 3.2 are satisfied with $\alpha_0 = 1$. In problem (3.11)-(3.15) on $[0, 1]$, by definition $\tilde{\xi} = (\tilde{l}(0), \tilde{x}(0), \tilde{z}(0); \tilde{l}(1), \tilde{x}(1), \tilde{z}(1))$. Since $\tilde{L}$ is an affine function of $\tilde{\xi}$, we have $\tilde{L}_{\tilde{\xi}\tilde{\xi}} = 0$. Since $\alpha_0 = 1$ we have

$$\tilde{H}(\tilde{l}, \tilde{x}, \tilde{z}, \tilde{f}, \tilde{p}^l, \tilde{p}^x, \tilde{p}^z) = \tilde{p}^x \tilde{l} + \tilde{p}^z \tilde{l}(\varphi(\tilde{z}) + e_1 \tilde{f}) + \tilde{l}F(\tilde{x}, \tilde{z}, \tilde{f}).$$

Recall that $\tilde{H}_{\tilde{f}} = \tilde{p}^z \tilde{l} e_1 + \alpha_0 \tilde{l}(\tilde{f} - f^*(\tilde{x}))$. Consequently, $\tilde{H}_{\tilde{f}\tilde{f}} = \tilde{l}$. Since $\tilde{l} = l \geq a > 0$, the strengthened Legendre condition is satisfied.

Let us write down the definition of the critical cone $\tilde{\mathcal{C}}$. Equations in variations for the system

$$\frac{d\tilde{l}}{dt} = 0, \quad \frac{d\tilde{x}}{dt} = \tilde{l}, \quad \frac{d\tilde{z}}{dt} = \tilde{l}(\varphi(\tilde{z}(t)) + e_1 \tilde{f}(t))$$

at the point $\tilde{w}$ have the form

$$\delta \dot{\tilde{l}} = 0, \quad \delta \dot{\tilde{x}}(t) = \delta \tilde{l}, \quad \delta \dot{\tilde{z}}(t) = \tilde{l}(\varphi'(\tilde{z}(t))\delta \tilde{z}(t) + e_1 \delta \tilde{f}(t)) + (\varphi(\tilde{z}(t)) + e_1 \tilde{f}(t))\delta \tilde{l}.$$

The endpoint conditions $\tilde{x}(0) = 0$ and $K(\tilde{z}(0), \tilde{z}(1)) = 0$ imply the following conditions in the critical cone

$$\delta \tilde{x}(0) = 0, \quad K'(\tilde{\zeta})\delta \tilde{\zeta} = 0,$$

where $\tilde{\zeta} = (\tilde{z}(0), \tilde{z}(1))$, $\delta \tilde{\zeta} = (\delta \tilde{z}(0), \delta \tilde{z}(1))$.

Further, recall that

$$\frac{\tilde{c}_H}{\tilde{l}} = \alpha_a - \alpha_b.$$

The initial conditions $-\tilde{l}(0) + a \leq 0$ and $\tilde{l}(0) - b \leq 0$ imply:

- if $a < \tilde{l} < b$, i.e., these constraints are not active, then $\tilde{c}_H = 0$, and we have no conditions on $\delta \tilde{l}(0)$,

- if $a = \tilde{l}$ and, therefore, $\tilde{l} < b$, then the following conditions are satisfied
 $\delta\tilde{l}(0) \geq 0$, $\tilde{c}_H\delta\tilde{l}(0) = 0$,
- if $\tilde{l} = b$ and, therefore, $\tilde{l} > a$, then the following conditions are satisfied
 $\delta\tilde{l}(0) \leq 0$, $\tilde{c}_H\delta\tilde{l}(0) = 0$.

Consequently,

$$\begin{aligned} \tilde{\mathcal{C}} = \left\{ \delta\tilde{w} = (\delta\tilde{l}, \delta\tilde{x}, \delta\tilde{z}, \delta\tilde{f}) : \quad & \delta\dot{\tilde{l}} = 0, \quad \delta\dot{\tilde{x}}(t) = \delta\tilde{l}, \quad \delta\tilde{x}(0) = 0, \quad K'(\tilde{\zeta})\delta\tilde{\zeta} = 0, \quad c_H\delta\tilde{l}(0) = 0, \right. \\ & \delta\dot{\tilde{z}}(t) = \tilde{l}(\varphi'(\tilde{z}(t))\delta\tilde{z}(t) + e_1\delta\tilde{f}(t)) + (\varphi(\tilde{z}(t)) + e_1\tilde{f}(t))\delta\tilde{l}, \\ & \left. \tilde{l} = a \implies \delta\tilde{l}(0) \geq 0; \quad \tilde{l} = b \implies \delta\tilde{l}(0) \leq 0 \right\}. \end{aligned}$$

As stated in Remark 1, if Ω is positive definite on $\tilde{\mathcal{C}}$, then it is positive definite on $(-\tilde{\mathcal{C}})$. Only one of the two conditions $\tilde{l} = a$ or $\tilde{l} = b$ could be realized. Therefore, the conditions $\tilde{l} = a \implies \delta\tilde{l}(0) \geq 0$; $\tilde{l} = b \implies \delta\tilde{l}(0) \leq 0$ in the definition of $\tilde{\mathcal{C}}$ can be omitted. More precisely, we can replace $\tilde{\mathcal{C}}$ with a subspace

$$\begin{aligned} \tilde{\Sigma} = \left\{ \delta\tilde{w} = (\delta\tilde{l}, \delta\tilde{x}, \delta\tilde{z}, \delta\tilde{f}) : \quad & \delta\dot{\tilde{l}} = 0, \quad \delta\dot{\tilde{x}}(t) = \delta\tilde{l}, \quad \delta\tilde{x}(0) = 0, \quad K'(\tilde{\zeta})\delta\tilde{\zeta} = 0, \quad \tilde{c}_H\delta\tilde{l}(0) = 0, \right. \\ & \left. \delta\dot{\tilde{z}}(t) = \tilde{l}(\varphi'(\tilde{z}(t))\delta\tilde{z}(t) + e_1\delta\tilde{f}(t)) + (\varphi(\tilde{z}(t)) + e_1\tilde{f}(t))\delta\tilde{l} \right\}. \end{aligned}$$

Note that if $\tilde{c}_H \neq 0$, then in the definition of $\tilde{\Sigma}$ we have $\delta\tilde{l}(0) = 0$, which gives $\delta\tilde{l} = 0$, and this means that $\delta\tilde{x} = 0$. In this case,

$$\tilde{\Sigma} = \left\{ \delta\tilde{w} = (\delta\tilde{l}, \delta\tilde{x}, \delta\tilde{z}, \delta\tilde{f}) : \quad \delta\tilde{l} = 0, \quad \delta\tilde{x} = 0, \quad \delta\dot{\tilde{z}}(t) = \tilde{l}\varphi'(\tilde{z}(t))\delta\tilde{z}(t) + \tilde{l}e_1\delta\tilde{f}(t), \quad K'(\tilde{\zeta})\delta\tilde{\zeta} = 0 \right\}.$$

Let us write down the quadratic form $\tilde{\Omega}$. Since $\alpha_0 = 1$,

$$\begin{aligned} \tilde{H}_{\tilde{l}} &= \tilde{p}^x + \tilde{p}^z(\varphi(\tilde{z}) + e_1\tilde{f}) + F(\tilde{x}, \tilde{z}, \tilde{f}), \\ \tilde{H}_{\tilde{x}} &= \tilde{l}F_{\tilde{x}} = -\tilde{l}(\tilde{z} - z^*(\tilde{x}))^\top (z^*)'(\tilde{x}) - \tilde{l}(\tilde{f} - f^*(\tilde{x}))(f^*)'(\tilde{x}), \\ \tilde{H}_{\tilde{z}} &= \tilde{p}^z\tilde{l}\varphi'(\tilde{z}) + \tilde{l}(\tilde{z} - z^*(\tilde{x}))^\top, \\ \tilde{H}_{\tilde{f}} &= \tilde{p}^z\tilde{l}e + \tilde{l}F_{\tilde{f}}(\tilde{x}, \tilde{z}, \tilde{f}) = \tilde{p}^z\tilde{l}e_1 + \tilde{l}(\tilde{f} - f^*(\tilde{x})). \end{aligned}$$

Once again we emphasize that we consider z , $\tilde{z}$, z^* as column vectors, and p^z , $\tilde{p}^z$, $\tilde{H}_{\tilde{z}}$ as row vectors. Therefore, $(\tilde{z} - z^*(\tilde{x}))^\top (z^*)'(\tilde{x}) = \sum_i (\tilde{z}_i - z_i^*(\tilde{x}))(z_i^*)'(\tilde{x})$.

The second-order partial derivatives have the form

$$\begin{aligned}
\tilde{H}_{\tilde{l}\tilde{l}} &= 0, \\
\tilde{H}_{\tilde{l}\tilde{x}} &= \tilde{H}_{\tilde{x}\tilde{l}} = -(\tilde{z} - z^*(\tilde{x}))^\top (z^*)'(\tilde{x}) - (\tilde{f} - f^*(\tilde{x}))(f^*)'(\tilde{x}), \\
\tilde{H}_{\tilde{l}\tilde{z}} &= \tilde{H}_{\tilde{z}\tilde{l}}^\top = \tilde{p}^z \varphi'(\tilde{z}) + (\tilde{z} - z^*(\tilde{x}))^\top, \\
\tilde{H}_{\tilde{l}\tilde{f}} &= \tilde{H}_{\tilde{f}\tilde{l}} = \tilde{p}_1^z + \tilde{f} - f^*(\tilde{x}), \\
\tilde{H}_{\tilde{x}\tilde{x}} &= \tilde{l}[(z^*)'(\tilde{x})]^\top (z^*)'(\tilde{x}) - \tilde{l}(\tilde{z} - z^*(\tilde{x}))^\top (z^*)''(\tilde{x}) \\
&\quad + \tilde{l}[(f^*)'(\tilde{x})]^2 - \tilde{l}(\tilde{f} - f^*(\tilde{x}))(f^*)''(\tilde{x}), \\
\tilde{H}_{\tilde{x}\tilde{z}} &= \tilde{H}_{\tilde{z}\tilde{x}}^\top = -\tilde{l}[(z^*)'(\tilde{x})]^\top, \\
\tilde{H}_{\tilde{x}\tilde{f}} &= \tilde{H}_{\tilde{f}\tilde{x}} = -\tilde{l}(f^*)'(\tilde{x}), \\
\tilde{H}_{\tilde{z}\tilde{z}} &= \tilde{p}^z \tilde{l} \varphi'(\tilde{z})^\top + \tilde{l} I_n, \\
\tilde{H}_{\tilde{z}\tilde{f}} &= \tilde{H}_{\tilde{f}\tilde{z}}^\top = 0, \\
\tilde{H}_{\tilde{f}\tilde{f}} &= \tilde{l}.
\end{aligned} \tag{3.22}$$

Here I_n is the identity matrix of size n and

$$(\tilde{z} - z^*(\tilde{x}))^\top (z^*)''(\tilde{x}) = \sum_i (\tilde{z}_i - z_i^*(\tilde{x}))(z_i^*)''(\tilde{x}).$$

Denoting $\tilde{w} = (\tilde{l}, \tilde{x}, \tilde{z}, \tilde{f})$, we get

$$\begin{aligned}
\langle \tilde{H}_{\tilde{w}\tilde{w}}(\tilde{l}, \tilde{x}, \tilde{z}, \tilde{f}, \tilde{p}^l, \tilde{p}^x, \tilde{p}^z) \delta \tilde{w}, \delta \tilde{w} \rangle &= \tilde{H}_{\tilde{l}\tilde{l}}(\delta \tilde{l})^2 + \tilde{H}_{\tilde{x}\tilde{x}}(\delta \tilde{x})^2 + \langle \tilde{H}_{\tilde{z}\tilde{z}} \delta \tilde{z}, \delta \tilde{z} \rangle + \tilde{H}_{\tilde{f}\tilde{f}}(\delta \tilde{f})^2 \\
&+ 2\tilde{H}_{\tilde{l}\tilde{x}} \delta \tilde{x} \cdot \delta \tilde{l} + 2\tilde{H}_{\tilde{l}\tilde{z}} \delta \tilde{z} \cdot \delta \tilde{l} + 2\tilde{H}_{\tilde{l}\tilde{f}} \delta \tilde{f} \cdot \delta \tilde{l} + 2\tilde{H}_{\tilde{x}\tilde{z}} \delta \tilde{z} \cdot \delta \tilde{x} + 2\tilde{H}_{\tilde{x}\tilde{f}} \delta \tilde{f} \cdot \delta \tilde{x} + 2\tilde{H}_{\tilde{z}\tilde{f}} \delta \tilde{z} \cdot \delta \tilde{f}.
\end{aligned}$$

Using the above formulas, we obtain

$$\begin{aligned}
&\langle \tilde{H}_{\tilde{w}\tilde{w}}(\tilde{l}(t), \tilde{x}(t), \tilde{z}(t), \tilde{f}(t), \tilde{p}^l(t), \tilde{p}^x(t), \tilde{p}^z(t)) \delta \tilde{w}(t), \delta \tilde{w}(t) \rangle \\
&= \tilde{l} \left([(z^*)'(\tilde{x}(t))]^\top (z^*)'(\tilde{x}(t)) - (\tilde{z}(t) - z^*(\tilde{x}(t)))^\top (z^*)''(\tilde{x}(t)) \right. \\
&\quad \left. + [(f^*)'(\tilde{x}(t))]^2 - (\tilde{f}(t) - f^*(\tilde{x}(t)))(f^*)''(\tilde{x}(t)) \right) (\delta \tilde{x}(t))^2 \\
&\quad + \tilde{l} \langle (\tilde{p}^z(t) \varphi''(\tilde{z}(t)) + I_n) \delta \tilde{z}(t), \delta \tilde{z}(t) \rangle + \tilde{l} (\delta \tilde{f}(t))^2 \\
&\quad - 2 \left((\tilde{z}(t) - z^*(\tilde{x}(t)))(z^*)'(\tilde{x}(t)) + (\tilde{f}(t) - f^*(\tilde{x}(t)))(f^*)'(\tilde{x}(t)) \right) \delta \tilde{x}(t) \cdot \delta \tilde{l} \\
&\quad + 2(\tilde{p}^z(t) \varphi'(\tilde{z}(t)) + (\tilde{z}(t) - z^*(\tilde{x}(t)))^\top) \delta \tilde{z}(t) \cdot \delta \tilde{l} \\
&\quad + 2(\tilde{p}_1^z(t) + \tilde{f}(t) - f^*(\tilde{x}(t))) \delta \tilde{f}(t) \cdot \delta \tilde{l} \\
&\quad - 2\tilde{l} \cdot [(z^*)'(\tilde{x}(t))]^\top \delta \tilde{z}(t) \cdot \delta \tilde{x}(t) - 2\tilde{l} \cdot (f^*)'(\tilde{x}(t)) \delta \tilde{f}(t) \cdot \delta \tilde{x}(t).
\end{aligned}$$

Recall that here $\tilde{l} = l = \text{const} > 0$. Since $\tilde{L}_{\tilde{\xi}\tilde{\xi}} = 0$, the quadratic form $\tilde{\Omega}$ is:

$$\tilde{\Omega}(\delta \tilde{w}) = \int_0^1 \langle \tilde{H}_{\tilde{w}\tilde{w}}(\tilde{l}, \tilde{x}, \tilde{z}, \tilde{f}, \tilde{p}^l, \tilde{p}^x, \tilde{p}^z) \delta \tilde{w}, \delta \tilde{w} \rangle dt.$$

Thus, we obtain the following result: *if there exists a constant $\tilde{c}_\Omega > 0$ such that*

$$\tilde{\Omega}(\delta\tilde{w}) \geq \tilde{c}_\Omega((\delta\tilde{l})^2 + |\delta\tilde{z}(0)|^2 + \|\delta\tilde{f}\|_2^2) \quad \forall \delta\tilde{w} \in \tilde{\Sigma},$$

then the quadruple $(\tilde{l}(\cdot), \tilde{x}(\cdot), \tilde{z}(\cdot), \tilde{f}(\cdot))$ is a weak local minimum in problem (3.11)-(3.15) on $[0, 1]$.

Now let us rewrite the obtained sufficient second-order condition in terms of the independent variable $x \in [0, l]$. Let $\delta\tilde{w} = (\delta\tilde{l}, \delta\tilde{x}, \delta\tilde{z}, \delta\tilde{f}) \in \tilde{\Sigma}$. Introduce $\delta z(x)$ such that $\delta z(\tilde{x}(t)) = \delta z(lt) = \delta\tilde{z}(t)$, that is $\delta z(x) = \delta\tilde{z}\left(\frac{x}{l}\right)$. Then $\delta\dot{\tilde{z}}(t) = (\delta z)'(\tilde{x}(t))l$, where $(\delta z)'(x) = \frac{dz(x)}{dx}$. Define δl such that $\delta\tilde{l} = l\delta l$, that is $\delta l = \frac{\delta\tilde{l}}{l}$. Similarly, we define $\delta f(x) = \delta\tilde{f}\left(\frac{x}{l}\right)$, $\delta x(x) = \delta\tilde{x}\left(\frac{x}{l}\right)$. Then the equation $\delta\dot{\tilde{z}}(t) = \tilde{l}(\varphi'(\tilde{z}(t))\delta\tilde{z}(t) + e_1\delta\tilde{f}(t)) + (\varphi(\tilde{z}(t)) + e_1\tilde{f}(t))\delta\tilde{l}$ takes the form

$$(\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1\delta f(x) + (\varphi(z(x)) + e_1f(x))\delta l$$

and the subspace $\tilde{\Sigma}$ in the new variables reads as follows

$$\Sigma = \left\{ \delta w = (\delta l, \delta x, \delta z, \delta f) : \begin{aligned} &(\delta l)' = 0, \quad (\delta x)'(x) = \delta l, \quad \delta x(0) = 0, \quad K'(\zeta)\delta\zeta = 0, \quad c_H\delta l(0) = 0, \\ &(\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1\delta f(x) + (\varphi(z(x)) + e_1f(x))\delta l \end{aligned} \right\},$$

where $\zeta = (z(0), z(l))$, $\delta\zeta = (\delta z(0), \delta z(l))$. Recall that δx and δl are one-dimensional, $\delta l = \text{const}$ and $\delta x = x\delta l$. Therefore,

$$\Sigma = \left\{ \delta w = (\delta l, \delta x, \delta z, \delta f) : \begin{aligned} &(\delta l)' = 0, \quad \delta x(x) = x \cdot \delta l, \quad K'(\zeta)\delta\zeta = 0, \quad c_H\delta l(0) = 0, \\ &(\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1\delta f(x) + (\varphi(z(x)) + e_1f(x))\delta l \end{aligned} \right\}.$$

Let us rewrite the quadratic form $\tilde{\Omega}$ in the new variables. Recall that $\tilde{l} = l$. If $x = \tilde{x}(t) = lt$, then $dx = ldt$ and $\tilde{z}(t) = z(x)$, $\tilde{f}(t) = f(x)$, $\delta\tilde{l} = l\delta l$, $\delta z(x) = \delta\tilde{z}(t)$, $\delta x(x) = \delta\tilde{x}(t)$,

$\delta f(x) = \delta \tilde{f}(t)$. Therefore, we have

$$\begin{aligned}
& \tilde{l} \left([(z^*)'(\tilde{x}(t))]^T (z^*)'(\tilde{x}(t)) - (\tilde{z}(t) - z^*(\tilde{x}(t)))^\top (z^*)''(\tilde{x}(t)) \right. \\
& \quad \left. + [(f^*)'(\tilde{x}(t))]^2 - (\tilde{f}(t) - f^*(\tilde{x}(t)))(f^*)''(\tilde{x}(t))) (\delta \tilde{x}(t))^2 dt \right. \\
& = \left([(z^*)'(x)]^\top (z^*)'(x) - (z(x) - z^*(x))^\top (z^*)''(x) \right. \\
& \quad \left. + [(f^*)'(x)]^2 - (f(x) - f^*(x))(f^*)''(x)) (\delta x(x))^2 dx, \right. \\
& \langle (\tilde{p}^z(t) \tilde{l} \varphi''(\tilde{z}(t)) + \tilde{l} I_n) \delta \tilde{z}(t), \delta \tilde{z}(t) \rangle dt = \langle (p^z(x) \varphi''(z(x)) + I_n) \delta z(x), \delta z(x) \rangle dx, \\
& l(\delta \tilde{f}(t))^2 dt = (\delta f(x))^2 dx, \\
& - 2 \left((\tilde{z}(t) - z^*(\tilde{x}(t)))^\top (z^*)'(\tilde{x}(t)) + (\tilde{f}(t) - f^*(\tilde{x}(t)))(f^*)'(\tilde{x}(t)) \right) \delta \tilde{x}(t) \cdot \delta \tilde{l} \cdot dt \\
& \quad = -2 \left((z(x) - z^*(x))^\top (z^*)'(x) + (f(x) - f^*(x))(f^*)'(x) \right) \delta x(x) \cdot \delta l \cdot dx, \\
& 2 \left((\tilde{p}^z(t) \varphi'(\tilde{z}(t)) + (\tilde{z}(t) - z^*(\tilde{x}(t)))^\top \right) \delta \tilde{z}(t) \cdot \delta \tilde{l} \cdot dt \\
& \quad = 2 \left((p^z(x) \varphi'(z(x)) + (z(x) - z^*(x))^\top \right) \delta z(x) \cdot \delta l \cdot dx, \\
& - 2 \tilde{l} [(z^*)'(\tilde{x}(t))]^\top \delta \tilde{z}(t) \cdot \delta \tilde{x}(t) \cdot dt - 2 \tilde{l} (f^*)'(\tilde{x}(t)) \cdot \delta \tilde{f}(t) \cdot \delta \tilde{x}(t) \cdot dt \\
& \quad = -2 [(z^*)'(x)]^\top \delta z(x) \cdot \delta x(x) \cdot dx - 2 (f^*)'(x) \delta f(x) \cdot \delta x(x) \cdot dx.
\end{aligned}$$

Consequently,

$$\tilde{\Omega}(\delta \tilde{w}) = \Omega(\delta w),$$

where

$$\begin{aligned}
\Omega(\delta w) = & \int_0^l \left\{ [(z^*)'(x)]^\top (z^*)'(x) - (z(x) - z^*(x))^\top (z^*)''(x) \right. \\
& + [(f^*)'(x)]^2 - (f(x) - f^*(x))(f^*)''(x)) (\delta x(x))^2 \\
& + \langle (p^z(x) \varphi''(z) + I_n) \delta z(x), \delta z(x) \rangle + (\delta f(x))^2 \\
& - 2 \left((z(x) - z^*(x))^\top (z^*)'(x) + (f(x) - f^*(x))(f^*)'(x) \right) \delta x(x) \cdot \delta l \\
& + 2 \left(p^z(x) \varphi'(z(x)) + (z(x) - z^*(x))^\top \right) \delta z(x) \cdot \delta l \\
& + 2(p_1^z(x) + f(x) - f^*(x)) \delta f(x) \cdot \delta l \\
& \left. - 2 [(z^*)'(x)]^\top \delta z(x) \cdot \delta x(x) - 2 (f^*)'(x) \delta f(x) \cdot \delta x(x) \right\} dx.
\end{aligned}$$

Below we replace p^z with p , omitting the superscript z . Since $\delta l \in \mathbb{R}$ is a constant,

$\delta x = x \cdot \delta l$, and δf is one-dimensional, we obtain

$$\begin{aligned}
\Omega(\delta w) = & (\delta l)^2 \int_0^l \left([(z^*)'(x)]^\top (z^*)'(x) - (z(x) - z^*(x))^\top (z^*)''(x) \right. \\
& + [(f^*)'(x)]^2 - (f(x) - f^*(x))(f^*)''(x) \Big) x^2 dx \\
& + \int_0^l \langle (p(x)\varphi''(z(x)) + I_n)\delta z(x), \delta z(x) \rangle dx + \int_0^l (\delta f(x))^2 dx \\
& - 2(\delta l)^2 \int_0^l \left((z(x) - z^*(x))^\top (z^*)'(x) + (f(x) - f^*(x))(f^*)'(x) \right) x dx \\
& + 2\delta l \int_0^l \left(p(x)\varphi'(z(x)) + (z(x) - z^*(x))^\top \right) \delta z(x) dx \\
& + 2\delta l \int_0^l (p_1(x) + f(x) - f^*(x)) \delta f(x) dx \\
& - 2\delta l \int_0^l ((z^*)'(x))^\top \delta z(x) x dx \\
& - 2\delta l \int_0^l (f^*)'(x) \delta f(x) x dx.
\end{aligned} \tag{3.23}$$

This quadratic form is independent of δx . We can exclude δx from the definition of Σ as well. Therefore, the quadratic form Ω is considered on a subspace, which we still denote by Σ (we also keep the notation δw for the shorter collection $(\delta l, \delta z, \delta f)$):

$$\begin{aligned}
\Sigma := \Big\{ \delta w = (\delta l, \delta z, \delta f) : & \quad (\delta l)' = 0, \quad c_H \delta l = 0, \quad K'(\zeta) \delta \zeta = 0, \\
& \quad (\delta z)'(x) = \varphi'(z(x)) \delta z(x) + e_1 \delta f(x) + (\varphi(z(x)) + e_1 f(x)) \delta l \Big\}.
\end{aligned}$$

Thus, we obtain the following result:

Theorem 6. *Let an admissible triple $(l, z(\cdot), f(\cdot))$ satisfy the first order necessary optimality conditions of Theorem 5 in problem (3.7)-(3.9) with the corresponding multipliers $\alpha_0 = 1, \beta, p(\cdot)$. Suppose there exists a constant $c_\Omega > 0$ such that*

$$\Omega(\delta w) \geq c_\Omega \left((\delta l)^2 + |\delta z(0)|^2 + \|\delta f\|_2^2 \right) \quad \forall \delta w \in \Sigma.$$

Then the triple $(l, z(\cdot), f(\cdot))$ is a weak local minimum in problem (3.7)-(3.9).

Matrix Riccati equation for One Beam: Case $C_H \neq 0$

In this case, as we know, the condition $C_H > 0$ implies $l = a$, and the condition $C_H < 0$ implies $l = b$. Then in the definition of Σ , we have $\delta l = 0$, so that we can put

$$\Sigma := \left\{ \delta w = (\delta z, \delta f) : (\delta z)'(x) = \varphi'(z(x)) \delta z(x) + e_1 \delta f(x), K'(\zeta) \delta \zeta = 0 \right\}.$$

Since $\delta l = 0$, the quadratic form reduces to

$$\Omega(\delta w) = \int_0^l \langle (p(x)\varphi''(z(x)) + I_n) \delta z(x), \delta z(x) \rangle dx + \int_0^l (\delta f(x))^2 dx.$$

We study the question of the positive definiteness of Ω on Σ in terms of the solution of the matrix Riccati equation. Obviously, the strengthened Legendre condition is satisfied.

Comparing the differential equation in the definition of Σ with the equation $(\delta z)' = A\delta z + B\delta f$ (see end of section 3.1.3), we obtain

$$A = \varphi'((z(x))), \quad B = e_1 = (1, 0, \dots, 0)^\top.$$

Comparing Ω with (3.4), we get

$$R = p(x)\varphi''(z(x)) + I_n, \quad S = 0, \quad U = 1.$$

Consequently,

$$\begin{aligned} (S + QB)U^{-1}(S^\top + B^\top Q) &= Qe_1e_1^\top Q = \begin{pmatrix} Q_{11} \\ \dots \\ Q_{1n} \end{pmatrix} (Q_{11} \quad \dots \quad Q_{1n}) \\ &= \begin{pmatrix} Q_{11}Q_{11} & \dots & Q_{11}Q_{1n} \\ \dots & \dots & \dots \\ Q_{1n}Q_{11} & \dots & Q_{1n}Q_{1n} \end{pmatrix} = \|Q_{1i}Q_{1j}\|_{i,j=1}^n. \end{aligned}$$

Thus, the Riccati equation (3.5) reduces to the following

$$\frac{d}{dx}Q + QA + A^\top Q + R - Qe_1e_1^\top Q = 0, \quad x \in [0, l]. \quad (3.24)$$

where

$$\begin{aligned} A &= \varphi'((z(x))), & R &= p(x)\varphi''(z(x)) + I_n, \\ e_1 &= (1, 0, \dots, 0)^\top, & Qe_1e_1^\top Q &= \|Q_{1i}Q_{1j}\|_{i,j=1}^n. \end{aligned}$$

The matrix M has the form

$$M = \begin{pmatrix} Q(0) & 0 \\ 0 & -Q(l) \end{pmatrix}.$$

To this Riccati equation, one can add the initial condition

$$Q(0) = I_n,$$

where I_n is the identity matrix of order n .

Similarly to Theorem 4 the following theorem holds.

Theorem 7. Assume that the strengthened Legendre condition is satisfied, $C_H \neq 0$, and there exists a symmetric solution Q (with the entries belonging to C^1) of the Riccati matrix equation (3.24) on $[0, l]$ such that

- (a) the matrix M is non-negative definite;
- (b) for all $\zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^{2n}$ the conditions $K'(\zeta)\zeta = 0$, $\langle M\zeta, \zeta \rangle = 0$ imply that $\zeta_0 = 0$ or $\zeta_1 = 0$.

Then the quadratic form Ω is positive definite on the subspace Σ .

Matrix Riccati equation for one beam: case $C_H = 0$

In this more complicated case, we have

$$\Sigma := \left\{ \delta w = (\delta l, \delta z, \delta f) : \begin{aligned} &(\delta l)' = 0, \quad K'(\zeta)\delta\zeta = 0, \\ &(\delta z)'(x) = \varphi'(z(x))\delta z(x) + e_1\delta f(x) + (\varphi(z(x)) + e_1f(x))\delta l \end{aligned} \right\}.$$

Consider again the sufficient condition for the positive definiteness of the quadratic form Ω on the subspace Σ . Now Σ is defined by a linear system of differential equations

$$\begin{cases} (\delta l)' = 0, \\ (\delta z)' = (\varphi(z(x)) + e_1f(x))\delta l + \varphi'(z(x))\delta z(x) + e_1\delta f(x). \end{cases}$$

In the sequel, we denote

$$\begin{aligned} X &= \begin{pmatrix} l \\ z \end{pmatrix} = \begin{pmatrix} l \\ z_1 \\ \dots \\ z_n \end{pmatrix} \in \mathbb{R}^{n+1}, \quad \delta X = \begin{pmatrix} \delta l \\ \delta z \end{pmatrix} = \begin{pmatrix} \delta l \\ \delta z_1 \\ \dots \\ \delta z_n \end{pmatrix} \in \mathbb{R}^{n+1}, \\ w &= \begin{pmatrix} X \\ f \end{pmatrix} = \begin{pmatrix} l \\ z \\ f \end{pmatrix} \in \mathbb{R}^{n+2}, \quad \delta w = \begin{pmatrix} \delta X \\ \delta f \end{pmatrix} = \begin{pmatrix} \delta l \\ \delta z \\ \delta f \end{pmatrix} \in \mathbb{R}^{n+2}. \end{aligned}$$

Let us represent the above system in matrix form $(\delta X)' = A\delta X + B\delta f$, where A is a $(n+1) \times (n+1)$ matrix, B is a $(n+1) \times 1$ matrix such that

$$A = \begin{pmatrix} 0 & 0_n^\top \\ \varphi(z(x)) + e_1f(x) & \varphi'(z(x)) \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ e_1 \end{pmatrix}, \quad 0_n^\top = (0, \dots, 0) \in \mathbb{R}^{n^\top}.$$

It is convenient to present

$$A := \begin{pmatrix} 0 & 0_n^\top \\ A_{zl} & A_{zz} \end{pmatrix}, \quad \text{where} \quad A_{zl} = \varphi(z(x)) + e_1f(x), \quad A_{zz} = \varphi'(z(x)).$$

Compare quadratic form (3.23) with the standard form (see (3.4)):

$$\Omega(\delta w) = \int_0^l \left(\langle R\delta X, \delta X \rangle + 2(\delta X)^\top S \delta f + U(\delta f)^2 \right) dt,$$

where R is the symmetric $(n+1) \times (n+1)$ matrix, $S \in \mathbb{R}^{n+1}$ is the column vector, U is the number. Let us find the matrix R . Denote

$$R = \begin{pmatrix} R_{ll} & R_{lz} \\ R_{zl} & R_{zz} \end{pmatrix},$$

where R_{ll} , R_{lz} , $R_{zl} = R_{lz}^\top$, $R_{zz} = R_{zz}^\top$ are matrices of orders 1×1 , $1 \times n$, $n \times 1$, $n \times n$, respectively. Then,

$$\langle R\delta X, \delta X \rangle = R_{ll}(\delta l)^2 + 2R_{lz}\delta z\delta l + \langle R_{zz}\delta z, \delta z \rangle.$$

Using (3.23), we obtain

$$\begin{aligned} \langle R\delta X, \delta X \rangle = & \left[\left([(z^*)'(x)]^\top (z^*)'(x) - (z(x) - z^*(x))^\top (z^*)''(x) \right. \right. \\ & + [(f^*)'(x)]^2 - (f(x) - f^*(x))(f^*)''(x) \Big) x^2 \\ & - 2 \left((z(x) - z^*(x))^\top (z^*)'(x) + (f(x) - f^*(x))(f^*)'(x) \right) x \Big] \cdot (\delta l)^2 \\ & + \langle (p(x)\varphi''(z(x)) + I_n) \delta z(x), \delta z(x) \rangle \\ & + 2 \left[(p(x)\varphi'(z(x)) + (z(x) - z^*(x))^\top - x((z^*)'(x))^\top \right] \delta z(x) \cdot \delta l. \end{aligned}$$

Consequently,

$$\begin{aligned} R_{ll} = & \left([(z^*)'(x)]^\top (z^*)'(x) - (z(x) - z^*(x))^\top (z^*)''(x) \right. \\ & + [(f^*)'(x)]^2 - (f(x) - f^*(x))(f^*)''(x) \Big) x^2 \\ & - 2 \left((z(x) - z^*(x))^\top (z^*)'(x) \right. \\ & \left. + (f(x) - f^*(x))(f^*)'(x) \right) x, \end{aligned} \quad (3.25)$$

$$R_{zz} = p(x)\varphi''(z(x)) + I_n, \quad (3.26)$$

$$R_{lz} = R_{zl}^\top = (p(x)\varphi'(z(x)) + (z(x) - z^*(x))^\top - x((z^*)'(x))^\top). \quad (3.27)$$

Further, $U = 1$, and finally, S has the form

$$S = \begin{pmatrix} S_l \\ 0_n \end{pmatrix} \in \mathbb{R}^{n+1},$$

where $S_l = p_1(x) + f(x) - f^*(x) - x(f^*)'(x)$. Recall that the Riccati equation has the form

$$\frac{d}{dx}Q + QA + A^\top Q + R - (S + QB)U^{-1}(S^\top + B^\top Q) = 0, \quad x \in [0, l],$$

where

$$Q(x) = \begin{pmatrix} Q_{ll} & Q_{lz} \\ Q_{zl} & Q_{zz} \end{pmatrix} (x)$$

where

$$Q_{ll}(x) \in \mathbb{R}, \quad Q_{zl}(x) = \begin{pmatrix} Q_{z_1 l} \\ \vdots \\ Q_{z_n l} \end{pmatrix} (x) \in \mathbb{R}^n,$$

$$Q_{lz}(x) = Q_{zl}^\top(x) = \begin{pmatrix} Q_{lz_1} & \cdots & Q_{lz_n} \end{pmatrix} (x) \in \mathbb{R}^{n^\top},$$

and

$$Q_{zz}(x) = \begin{pmatrix} Q_{z_1 z_1} & \cdots & Q_{z_1 z_n} \\ \vdots & \ddots & \vdots \\ Q_{z_n z_1} & \cdots & Q_{z_n z_n} \end{pmatrix} (x)$$

is $n \times n$ symmetric matrix. Since $U = 1$, we have

$$(S + QB)U^{-1}(S^\top + B^\top Q) = (S + QB)(S + QB)^\top.$$

Further,

$$QB = \begin{pmatrix} Q_{ll} & Q_{lz} \\ Q_{zl} & Q_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ e_1 \end{pmatrix} = \begin{pmatrix} Q_{lz}e_1 \\ Q_{zz}e_1 \end{pmatrix}.$$

Hence

$$S + QB = \begin{pmatrix} S_l \\ 0 \end{pmatrix} + \begin{pmatrix} Q_{lz}e_1 \\ Q_{zz}e_1 \end{pmatrix} = \begin{pmatrix} Q_{lz}e_1 + S_l \\ Q_{zz}e_1 \end{pmatrix}.$$

Consequently,

$$\begin{aligned} (S + QB)(S + QB)^\top &= \begin{pmatrix} Q_{lz}e_1 + S_l \\ Q_{zz}e_1 \end{pmatrix} (Q_{lz}e_1 + S_l, e_1^\top Q_{zz}) \\ &= \begin{pmatrix} (Q_{lz}e_1 + S_l)^2(Q_{lz}e_1 + S_l)e_1^\top Q_{zz} & \\ Q_{zz}e_1(Q_{lz}e_1 + S_l) & (Q_{zz}e_1)(e_1^\top Q_{zz}) \end{pmatrix}. \end{aligned}$$

Moreover,

$$QA = \begin{pmatrix} Q_{ll} & Q_{lz} \\ Q_{zl} & Q_{zz} \end{pmatrix} \begin{pmatrix} 0 & 0_n^\top \\ A_{zl} & A_{zz} \end{pmatrix} = \begin{pmatrix} Q_{lz}A_{zl} & Q_{lz}A_{zz} \\ Q_{zz}A_{zl} & Q_{zz}A_{zz} \end{pmatrix}, \quad A^\top Q = \begin{pmatrix} Q_{lz}A_{zl} & A_{zl}^\top Q_{zz} \\ A_{zz}^\top Q_{zl} & A_{zz}^\top Q_{zz} \end{pmatrix}.$$

Here $Q_{lz}A_{zl}$, $Q_{lz}A_{zz}$, $Q_{zz}A_{zl}$, $Q_{zz}A_{zz}$ are matrices of order 1×1 , $1 \times n$, $n \times 1$, $n \times n$, respectively. Consequently,

$$QA + A^\top Q = \begin{pmatrix} 2Q_{lz}A_{zl} & Q_{lz}A_{zz} + A_{zl}^\top Q_{zz} \\ Q_{zz}A_{zl} + A_{zz}^\top Q_{zl} & Q_{zz}A_{zz} + A_{zz}^\top Q_{zz} \end{pmatrix}.$$

Thus, according to (3.5), we obtain the matrix Riccati equation in the form

$$\begin{aligned} \frac{dd}{ddx} \begin{pmatrix} Q_{ll} & Q_{lz} \\ Q_{zl} & Q_{zz} \end{pmatrix} &+ \begin{pmatrix} 2Q_{lz}A_{zl} & Q_{lz}A_{zz} + A_{zl}^\top Q_{zz} \\ Q_{zz}A_{zl} + A_{zz}^\top Q_{zl} & Q_{zz}A_{zz} + A_{zz}^\top Q_{zz} \end{pmatrix} + \begin{pmatrix} R_{ll} & R_{lz} \\ R_{zl} & R_{zz} \end{pmatrix} \\ &- \begin{pmatrix} (Q_{lz}e_1 + S_l)^2 & (Q_{lz}e_1 + S_l)e_1^\top Q_{zz} \\ (Q_{lz}e_1 + S_l)Q_{zz}e_1 & (Q_{zz}e_1)(e_1^\top Q_{zz}) \end{pmatrix} = 0, \quad x \in [0, l], \end{aligned} \quad (3.28)$$

where the blocks of the matrix R are determined by formulas (3.25)-(3.27). Further, the matrix M has the form

$$M = \begin{pmatrix} Q(0) & 0 \\ 0 & -Q(l) \end{pmatrix}.$$

We set

$$\delta\xi = \begin{pmatrix} \delta X(0) \\ \delta X(l) \end{pmatrix}.$$

Then

$$\langle M\delta\xi, \delta\xi \rangle = \langle Q(0)\delta X(0), \delta X(0) \rangle - \langle Q(l)\delta X(l), \delta X(l) \rangle,$$

where

$$\delta X(0) = \begin{pmatrix} \delta l \\ \delta z(0) \end{pmatrix}, \quad \delta X(l) = \begin{pmatrix} \delta l \\ \delta z(l) \end{pmatrix}, \quad (\delta l)' = 0, \text{ i.e., } \delta l = \text{const.}$$

The condition $\mathcal{E}\delta\xi = 0$ in the definition of Σ means $K_{z_0}\delta z(0) + K_{z_l}\delta z(l) = 0$, $(\delta l)' = 0$. Consequently,

$$\delta X_0 := \delta X(0) = \begin{pmatrix} \delta l \\ \delta z_0 \end{pmatrix}, \quad \delta X_l := \delta X(l) = \begin{pmatrix} \delta l \\ \delta z_l \end{pmatrix}.$$

The following theorem holds.

Theorem 8. *Assume that the strengthened Legendre condition is satisfied, $C_H = 0$, and there exists a symmetric solution Q (with the entries belonging to C^1) of the Riccati matrix equation (3.28) on $[0, l]$ such that*

- (a) *the matrix M is nonnegative definite;*
- (b) *for all pairs of vectors in $\mathbb{R}^{n+1}$*

$$\delta X_0 = \begin{pmatrix} \delta l \\ \delta z_0 \end{pmatrix}, \quad \delta X_l = \begin{pmatrix} \delta l \\ \delta z_l \end{pmatrix}$$

the conditions $K_{z_0}\delta z_0 + K_{z_l}\delta z_l = 0$, $\delta l \in \mathbb{R}$, $\langle Q(0)\delta X_0, \delta X_0 \rangle - \langle Q(l)\delta X_l, \delta X_l \rangle = 0$ imply that $\delta X_0 = 0$ or $\delta X_l = 0$. Then the quadratic form Ω is positive definite on the subspace Σ .

3.3 Numerical Examples

In the following examples, we focus on a single edge to show the application of our methods. In the first example, we construct a semilinear differential equation, specifying the desired state and control parameters. Subsequently, we derive the optimal state and control configurations through the optimality system.

For the second example, we consider the static IGEB model, with parameters detailed in [41]. First, we semi-discretize in space the IGEB model by means of the finite element method and using $\mathbb{P}_2$ (quadratic) elements. For the nonlinear ODE, we use the Newton–Raphson method.

3.3.1 Example 1

Consider a steady state scenario involving a single edge, governed by a semilinear differential equation. The control system has the form

$$z'(x) = \varphi(z(x)) + f(x), \quad x \in [0, l], \quad z(l) = 0,$$

where z is one dimensional and l is not fixed. The beam's behavior is described by the function

$$\varphi(z) = z - z^2.$$

Set

$$l^* = 1, \quad z^*(x) = -2 + x + x^2, \quad f^*(x) = 7 - 3x - 4x^2 + 2x^3 + x^4.$$

It is easy to check that the triple $(l^*, z^*, f^*(x))$ defines an admissible process of a given control system.

The cost functional is expressed as:

$$J = \frac{1}{2} \int_0^l ((z(x) - z^*(x))^2 + (f(x) - f^*(x))^2) dx + \frac{1}{2}(l - l^*)^2 \rightarrow \min.$$

The parameter l is constrained $l \in [\frac{1}{2}, \frac{3}{2}]$.

Obviously, $(l^*, z^*, f^*(x))$ is the solution to this problem. But assume that this solution is unknown and let us write down the necessary optimality conditions of Theorem 5.

Since $\int_0^l (x - l^*) dx = \frac{1}{2}(l - l^*)^2 - \frac{1}{2}(l^*)^2$, we can consider the equivalent problem of minimizing the functional

$$J = \int_0^l F(x, z(x), f(x)) dx$$

with $F(x, z, f) = \frac{1}{2}((z - z^*(x))^2 + (f - f^*(x))^2) + x - l^*$.

Let the triple $(l, z(\cdot), f(\cdot))$ be a solution to this problem. Then, according to Theorem 5 there are numbers $\alpha_0 \geq 0$, β , and a continuously differentiable function $p : [0, l] \rightarrow \mathbb{R}$ such that

$$\begin{aligned} z'(x) &= \varphi(z(x)) + f(x), \quad z(l) = 0 \\ -p'(x) &= p(x)\varphi'(z(x)) + \alpha_0 F_z(x, z(x), f(x)), \quad x \in [0, l] \\ p(0) &= 0, \quad p(l) = \beta, \\ p(x) + \alpha_0 F_f(x, z(x), f(x)) &= 0, \quad x \in [0, l]. \end{aligned}$$

If $\alpha_0 = 0$, then $p(x) = 0$ and $\beta = 0$. Therefore, $\alpha_0 > 0$, and we put $\alpha_0 = 1$. Hence, taking into account that $\varphi(z) = z - z^2$, $\varphi'(z) = 1 - 2z$, $F_z(x, z, f) = (z - z^*(x))$, and $F_f(x, z, f) = f - f^*(x)$, we get a system

$$\left. \begin{aligned} z'(x) &= z(x) - z^2(x) + f(x), \quad z(l) = 0, \\ -p'(x) &= p(x)(1 - 2z(x)) + z - z^*(x), \quad p(0) = 0 \\ p(x) + f(x) - f^*(x) &= 0. \end{aligned} \right\} \quad (3.29)$$

Theorem 5 gives one more necessary optimality condition for determining $(l, z(\cdot), f(\cdot))$. Recall that we are considering l close to $l^* = 1$, which means $a < l < b$ with $a = 0.5$, $b = 1.5$. As we know, in this case $c_H = 0$. Since $\alpha_0 = 1$, this condition looks like

$$c_H = p^x(x) + p(x)(\varphi(z(x)) + f(x)) + F(x, z(x), f(x)) = 0 \quad \forall x \in [0, l],$$

where $p^x(x) = \alpha_0 \int_x^l F_x(y, z(y), f(y)) dy$, $x \in [0, l]$. Considering that $p^x(l) = 0$, $z(l) = 0$, and $-p(l) = f(l) - f^*(l)$, we get

$$\begin{aligned} 0 &= p(l)(\varphi(z(l)) + f(l)) + F(l, z(l), f(l)) \\ &= p(l)(\varphi(0) + f(l)) + F(l, 0, f(l)) \\ &= p(l)f(l) + \frac{1}{2}(z^*(l))^2 + \frac{1}{2}((f(l) - f^*(l))^2 + l - 1 \\ &= p(l)f(l) + \frac{1}{2}(z^*(l))^2 + \frac{1}{2}(p(l))^2 + l - 1, \end{aligned}$$

that is

$$p(l)f(l) + \frac{1}{2}(z^*(l))^2 + \frac{1}{2}(p(l))^2 + l - 1 = 0. \quad (3.30)$$

Conditions (3.29) and (3.30) constitute a complete system of necessary optimality conditions for determining $(l, z(\cdot), f(\cdot))$. Obviously, the triple $p(x) = 0$, $f(x) = f^*(x)$, $z(x) = z^*(x)$ is a solution to this system.

We will now show numerical results for this problem. We conducted the computation using the finite element method and the Newton method to handle the nonlinear component. Here are the results. Fig. 3.1 illustrates the variation of the cost functional with respect to the length parameter. It is observed that the cost functional attains its minimum value at $l = 1 = l^*$, indicating the optimality of this length. This signifies that the length $l = 1$ is the optimal choice based on the minimization of the cost functional.

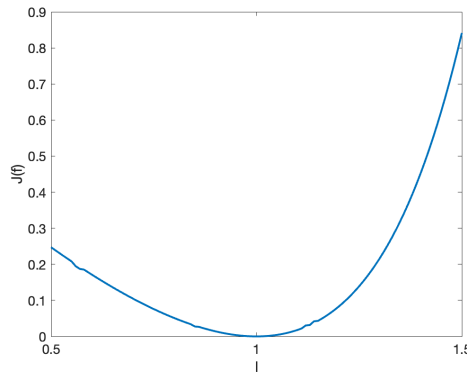


Figure 3.1: Cost with respect to l

In Fig. 3.2, we show the optimal control and state under the optimal length.

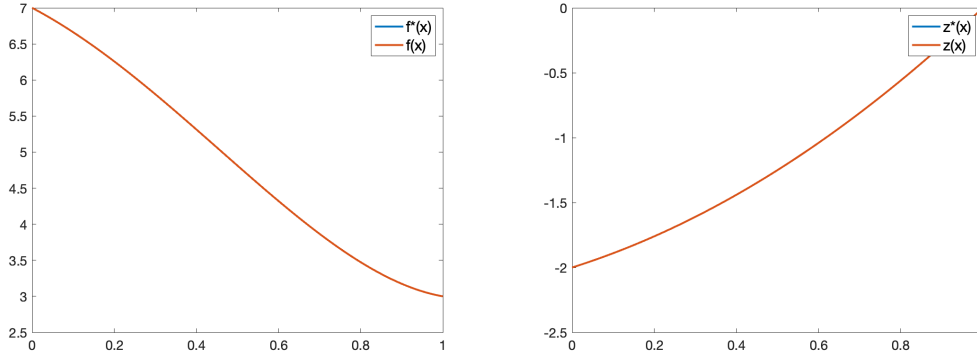


Figure 3.2: Optimal control and optimal state

Then, we computed the L^2 norm error between the numerical solution and the analytical solution to assess the accuracy of the results:

$$f_{err} = \|f(x) - f^*(x)\| = 1.6236e - 12, \quad z_{err} = \|z(x) - z^*(x)\| = 4.5426e - 11.$$

3.3.2 Example 2

Consider the steady state for a single beam governed by the semilinear differential equation (Eq. 2.7). We present results from numerical simulations. The flexibility matrices are given by

$$\mathbf{C} = \text{diag}(c_1, c_2, c_3, c_4, c_5, c_6)^{-1} = \text{diag}(10^4, 10^4, 10^4, 500, 500, 500)^{-1}.$$

In Eq. (3.7), setting $f^*(x) = -1$, then the steady state values are $z_1^* = -x + 1, z_2^* = 0, \dots, z_6^* = 0$. For IGEB model, the function

$$\varphi(r) = -\mathbf{E}(x)r + \mathbf{L}(r)\mathbf{C}r,$$

and its derivative is given by

$$\varphi'(r) = -\mathbf{E}(x) + (\mathbf{L}(r)\mathbf{C}r)' := -\mathbf{E}(x) + \bar{G}(r).$$

The optimality system of equations can be written as

$$\begin{cases} z'(x) = -\mathbf{E}(x)z(x) + \mathbf{L}(z)\mathbf{C}z(x) + e_1(f^* - p_1(x)), & x \in [0, l] \\ -p'(x) = -p(x)\mathbf{E}(x) + p(x)\bar{G}(z) + z(x) - z^*(x), & x \in [0, l] \\ z_i(l) = 0, & i = 1, 2, \dots, 6 \\ p_i(0) = 0, & i = 1, 2, \dots, 6. \end{cases}$$

The weak form of this system is given by:

$$\left\{ \begin{array}{l} -\int_0^l \left\langle \frac{d\psi}{dx}, z \right\rangle dx - z(0)\psi(0) + \int_0^l \langle \mathbf{E}z, \psi \rangle dx - \int_0^l \langle \mathbf{CL}(z)z, \psi \rangle dx \\ \quad + \int_0^l e_1 \langle p, \psi \rangle dx = \int_0^l e_1 \langle f^*, \psi \rangle dx, \forall \psi \in V_1 \\ \int_0^l \left\langle \frac{d\eta}{dx}, p \right\rangle dx - p(1)\eta(1) + \int_0^l \langle \mathbf{E}p, \eta \rangle dx - \int_0^l \langle \bar{G}(z)p, \eta \rangle dx \\ \quad - \int_0^l \langle z, \eta \rangle dx = - \int_0^l \langle z^*, \eta \rangle dx, \forall \eta \in V_2, \end{array} \right. \quad (3.31)$$

where

$$V_1 := \{\psi \in H^1(0, l; \mathbb{R}^6), \psi(1) = 0\},$$

and

$$V_2 := \{\eta \in H^1(0, l; \mathbb{R}^6), \eta(0) = 0\}.$$

In numerical discretization, the interval $[0, l]$ is discretized into N_x points $\{x_k\}_{k=1}^{N_x}$, where $x_1 = 0$ and $x_{N_x} = l$. Each subinterval $\omega^e := [x_{2e-1}, x_{2e+1}]$ for $e \in \{1, 2, \dots, N_e\}$ constitutes an element. These elements are defined by the points x_{2e-1} , x_{2e} , and x_{2e+1} and have a uniform length $h_e = x_{2e+1} - x_{2e-1}$. It is important to note that $N_x = 2N_e + 1$.

We utilize $\mathbb{P}_2$ (quadratic) elements to define function spaces $V_{1,h}$ and $V_{2,h}$ as described below:

$$\begin{aligned} V_{1,h} &:= \left\{ \psi \in C^0([0, l]; \mathbb{R}^{N_6}) : \psi|_{\omega^e} \in (\mathbb{P}_2)^{N_6} \text{ for all } e \in \{1, \dots, N_e\}, \psi(1) = 0 \right\}, \\ V_{2,h} &:= \left\{ \eta \in C^0([0, l]; \mathbb{R}^{N_6}) : \eta|_{\omega^e} \in (\mathbb{P}_2)^{N_6} \text{ for all } e \in \{1, \dots, N_e\}, \eta(0) = 0 \right\}. \end{aligned}$$

The approximations for $z_i(x)$ and $p_i(x)$ are represented by the following expressions:

$$z_i(x) = \sum_{j=1}^{N_x} Z_{i,j} \psi_j(x), \quad p_i(x) = \sum_{j=1}^{N_x} P_{i,j} \eta_j(x),$$

where $Z_{i,j}$ denotes the value of z_i at the $\mathbb{P}_2$ basis function ψ_j , which value is 1 at node x_j and 0 at other nodes, and similarly for $P_{i,j}$. In the discretized system, we define the following matrices and vectors:

$$\begin{aligned} A_1 &= \int_0^l \boldsymbol{\psi} \boldsymbol{\psi}^\top, \quad A_2 = \int_0^l \boldsymbol{\psi} (\boldsymbol{\psi}')^\top, \quad A_3[z] = \int_0^l z \boldsymbol{\psi} \boldsymbol{\psi}^\top, \\ \bar{A}_1 &= \int_0^l \boldsymbol{\eta} \boldsymbol{\eta}^\top, \quad \bar{A}_2 = \int_0^l \boldsymbol{\eta} (\boldsymbol{\eta}')^\top, \quad \bar{A}_3[z] = \int_0^l z \boldsymbol{\eta} \boldsymbol{\eta}^\top, \end{aligned}$$

where $\boldsymbol{\psi} = (\psi_1, \psi_2, \dots, \psi_{N_x})^\top$ and $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_{N_x})^\top$. The matrix form of system (3.31) can be written as

$$\begin{aligned} -K_{s,1}Z - M_{\mathbf{CL}(z)}Z + \bar{e}_1\bar{M}P &= \bar{e}_1\hat{F}, \\ K_{s,2}P - M_{G(z)}P - \bar{M}Z &= -\hat{Z}, \end{aligned} \quad (3.32)$$

i.e.,

$$\begin{pmatrix} -K_{s,1} & \bar{e}_1\bar{M} \\ -\bar{M} & K_{s,2} \end{pmatrix} \begin{pmatrix} Z \\ P \end{pmatrix} - \begin{pmatrix} M_{\mathbf{CL}(z)} & \\ & M_{G(z)} \end{pmatrix} \begin{pmatrix} Z \\ P \end{pmatrix} = \begin{pmatrix} \bar{e}_1\hat{F} \\ -\hat{Z} \end{pmatrix}, \quad (3.33)$$

where $\bar{e}_1 = \text{diag}(\mathbb{I}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x})$. $\mathbb{O}_{N_x}$ is the matrix of zeros. Furthermore, the vectors Z and P are defined as:

$$\begin{aligned} Z &= (Z_{1,1}, \dots, Z_{1,N_x}, \dots, Z_{6,1}, \dots, Z_{6,N_x})^\top, \\ P &= (P_{1,1}, \dots, P_{1,N_x}, \dots, P_{6,1}, \dots, P_{6,N_x})^\top. \end{aligned}$$

Similarity,

$$\begin{aligned} \hat{Z} &= (\hat{Z}_{1,1}, \dots, \hat{Z}_{1,N_x}, \dots, \hat{Z}_{6,1}, \dots, \hat{Z}_{6,N_x})^\top, \\ \hat{F} &= (\hat{F}_1, \dots, \hat{F}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x}, \mathbb{O}_{N_x})^\top, \end{aligned}$$

where $\hat{Z}_{i,j}$ represents the value of z_i^* at the basis function ψ_j and $\hat{F}_j$ represents the value of f^* at the basis function η_j .

The other matrices are defined by:

$$\begin{aligned} K_{s,1} &= \begin{pmatrix} A_2 & & & & & \\ & A_2 & & & & \\ & & A_2 & & & \\ & & & A_2 & & \\ & & & & A_2 & \\ -A_1 & & A_1 & & A_2 & \end{pmatrix}, \quad K_{s,2} = \begin{pmatrix} \bar{A}_2 & & & & & \\ & \bar{A}_2 & & & & \\ & & \bar{A}_2 & & & \\ & & & \bar{A}_2 & & \\ & & & -\bar{A}_1 & & \bar{A}_2 \\ \bar{A}_1 & & & & \bar{A}_2 & \bar{A}_2 \end{pmatrix}, \\ M_{G(z)} &= \begin{pmatrix} \mathbf{0} & c_6\bar{A}_3(z_6) & -c_5\bar{A}_3(z_5) & \mathbf{0} & -c_5\bar{A}_3(z_3) & c_6\bar{A}_3(z_2) \\ -c_6\bar{A}_3(z_6) & \mathbf{0} & c_4\bar{A}_3(z_4) & c_4\bar{A}_3(z_3) & \mathbf{0} & -c_6\bar{A}_3(z_1) \\ c_5\bar{A}_3(z_5) & -c_4\bar{A}_3(z_4) & \mathbf{0} & -c_4\bar{A}_3(z_2) & c_5\bar{A}_3(z_1) & \mathbf{0} \\ \mathbf{0} & (c_3 - c_2)\bar{A}_3(z_3) & (c_3 - c_2)\bar{A}_3(z_2) & \mathbf{0} & (c_6 - c_5)\bar{A}_3(z_6) & (c_6 - c_5)\bar{A}_3(z_5) \\ (c_1 - c_3)\bar{A}_3(z_3) & \mathbf{0} & (c_1 - c_3)\bar{A}_3(z_1) & (c_4 - c_6)\bar{A}_3(z_6) & \mathbf{0} & (c_4 - c_6)\bar{A}_3(z_4) \\ (c_2 - c_1)\bar{A}_3(z_2) & (c_2 - c_1)\bar{A}_3(z_1) & \mathbf{0} & (c_5 - c_4)\bar{A}_3(z_5) & (c_5 - c_4)\bar{A}_3(z_4) & \mathbf{0} \end{pmatrix}, \\ M_{\mathbf{CL}(z)} &= \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -c_5A_3(z_3) & c_6A_3(z_2) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & c_4A_3(z_3) & \mathbf{0} & -c_6A_3(z_1) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -c_4A_3(z_2) & c_5A_3(z_1) & \mathbf{0} \\ \mathbf{0} & -c_2A_3(z_3) & c_3A_3(z_2) & \mathbf{0} & -c_5A_3(z_6) & c_6A_3(z_5) \\ c_1A_3(z_3) & \mathbf{0} & -c_3A_3(z_1) & c_4A_3(z_6) & \mathbf{0} & c_6A_3(z_4) \\ -c_1A_3(z_2) & c_2A_3(z_1) & \mathbf{0} & -c_4A_3(z_5) & c_5A_3(z_4) & \mathbf{0} \end{pmatrix}, \end{aligned}$$

$$\bar{M} = \text{diag}(A_1, A_1, A_1, A_1, A_1, A_1).$$

Denote

$$A = \begin{pmatrix} -K_{s,1} & \bar{e}_1 \bar{M} \\ -\bar{M} & K_{s,2} \end{pmatrix}, N_L(z) = \begin{pmatrix} M_{\mathbf{CL}(z)} \\ M_{G(z)} \end{pmatrix}, W = \begin{pmatrix} Z \\ P \end{pmatrix}, \mathbf{F} = \begin{pmatrix} \bar{e}_1 \hat{F} \\ -\hat{Z} \end{pmatrix}.$$

So Eq. (3.33) becomes:

$$AW - N_L(z)W = \mathbf{F}, \quad (3.34)$$

where $N_L(z)$ represents the nonlinear component. The iterative process is governed by the equation:

$$AW^{[n+1]} - N_L(z^{[n]})W^{[n+1]} = \mathbf{F},$$

where the superscript $[n]$ denotes the n -th iteration. With an auxiliary function S_n :

$$S_n(\zeta) = A\zeta - N_L(z^{[n]})\zeta - \mathbf{F},$$

equation (3.34) becomes $S_n(W^{[n+1]}) = 0$. To find an approximate solution to $S_n(\zeta) = 0$, we employ the Newton-Raphson method, i.e., find ζ such that $S_n(\zeta) = 0$, by means of the scheme:

$$\zeta_{n+1} = \zeta_n - (\text{Jac } S_n(\zeta_n))^{-1} S_n(\zeta_n),$$

where $\text{Jac } S_n = A - N_L(z^{[n]})$.

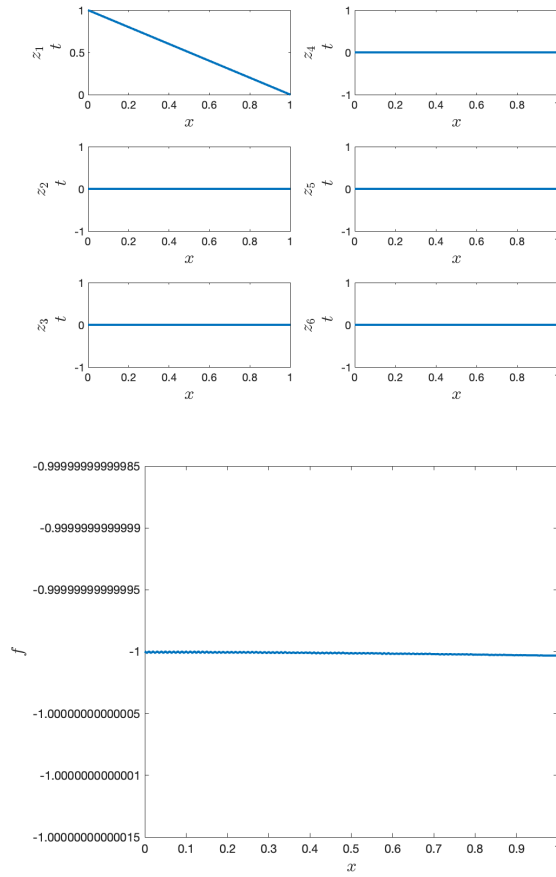
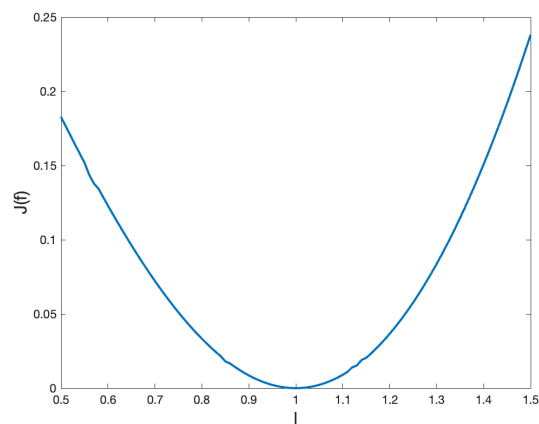
For our problem, the initial data is set to zero. The following Algorithm 1 outlines the steps taken to approximate the solution to Eq. (3.31).

Algorithm 1: Solve the obtained ODE for W

Set $\mathbf{C}, f^*, z^*$;
 Given initial guesses z^0 ;
while convergence **do**
 | $\zeta_{n+1} = \zeta_n - (\text{Jac } S_n(\zeta_n))^{-1} S_n(\zeta_n)$;
end
 $W = \zeta_{n+1}$

The results of this iterative scheme are visually presented in Fig. 3.3. These figures demonstrate that the optimal state and control closely approach z^* and f^* respectively when $l = l^*$.

Furthermore, Fig. 3.4 illustrates that the cost is convex with respect to the length of the beam with the unique minimizer. The optimal design corresponds to the length $l = l^* = 1$.

Figure 3.3: Optimal state z (top) and optimal force f (bottom) ($l = l^*$)Figure 3.4: Cost with respect to l

Chapter 4

Static Optimization for Timoshenko Beams

We start with the static model for Timoshenko beams. The optimal control problem is solved by using the optimality system. The cost for optimal design of networks is defined as the optimal cost for control problems.

4.1 Introduction

In this chapter, we investigate an optimal control problem with Timoshenko beams as an underlying physical model [61]. Optimal control is - roughly speaking - concerned with minimizing a certain cost functional that includes a partial differential equation as a constraint or as a design-to-state mapping where the state might be part of the arguments of the cost functional [40]. Optimum design problems Timoshenko beams are solved in [35]. Topological derivatives of the energy functional for static problems of Timoshenko beams are derived in [61].

We present new results on topological derivatives for general shape functionals for TB. The results are derived by using the domain decomposition techniques combined with the asymptotic analysis of the Steklov-Poincaré operator. The small parameter of the analysis is the size of the small cycle introduced in the network at some center nodes. In this way, the topology optimization of the network can be performed by nucleation of the small cycles.

Define $\Omega = \overline{\cup_{i \in \mathcal{I}} \Omega_i}$. We are interested in control problems defined on G . The controls are denoted by $v := v(x)$ with $x \in \Omega$, and by $u_T := u_T(x, t)$ with $x \in \Omega$ and $t \in (0, T)$. There are two control problems, the first is the static control problem, and the second is the evolution control problem. The static state equation gives the state $z := z(v; x) = z(x)$ that is determined by the state equation for

$$z \in H : a(z, \phi) = (L(v), \phi) + (f, \phi) \quad \forall \phi \in H, \quad (4.1)$$

where $x \mapsto z(v; x)$ lives in the Hilbert space H . In the applications usually H is a Sobolev

space. We assume that the bilinear form $a(\cdot, \cdot)$ is symmetric and coercive with a bounded linear operator L from the space of controls to H and $f \in H$.

We need as well the cost functional, the simplest possibility is the quadratic cost with the appropriate choice of norms in Hilbert function spaces. That is, we want to use a simple tracking-type cost functional for the state and velocity of a beam. For the static problem, it is

$$I(v) = \frac{1}{2} \|z - z^d\|_{L^2(\Omega)}^2 + \frac{1}{2} \|v - v^d\|_{L^2(\Gamma)}^2. \quad (4.2)$$

For the evolution problem, it is

$$J_T(u) = \frac{1}{2} \int_0^T \|y - y^d\|_{L^2(\Omega)}^2 + \gamma \|\partial_t(y - y^d)\|_{L^2(\Omega)}^2 dt + \frac{1}{2} \int_0^T \|u - u^d\|_{L^2(\Gamma)}^2 dt \quad (4.3)$$

where $\gamma \in (0, \infty)$. For the sake of simplicity, we consider the control problems without constraints.

The optimality conditions are necessary and sufficient and can be obtained by using the Lagrangian formalism. To this end, we need the adjoint states, which simplify the formulas for the gradients of the cost with respect to controls. We ask two questions now: the first is the shape and topology optimization for the graph with respect to the control problem, this results in the bi-level optimization problem for the graph. The second question is the turnpike property for the two-level optimization problem for the graph.

4.2 Optimality System for the Static Problem

We denote by $(\hat{z}, \hat{v}, \hat{p})$ the unique optimal solution of the static control problem. Let us note that the uniqueness of the optimal control $\hat{u}$ follows by the convexity of the variational problem under consideration with quadratic cost functional and the linear state equation. The optimal solution is given by the optimality system which depends on the *shape, or design* Ω of the network as an infinite dimensional factor or parameter to be selected at the upper level of optimization over the class of admissible shapes. The class of admissible shapes is denoted by S_{ad} , and the continuous variation of the shape is denoted by $\Omega_\tau \in S_{ad}$ with the real parameter $\tau \in (-\delta, \delta)$ for some $\delta > 0$ for shape variations. We use as well the singular perturbations of the shape denoted by $\Omega_\epsilon \in S_{ad}$ for the topology variations with $\epsilon \rightarrow 0$.

Remark 2. *In the case of a network, there are at least two possibilities of shape variations. The first is the change of the lengths of edges, it corresponds to the boundary variations in the classical shape optimization. The second, which corresponds to the topology variations in the shape optimization means the presence of a small cycle within the network with the small size of the cycle $\epsilon \rightarrow 0$.*

The optimality system for the static control problem is equivalent to the vanishing of the gradient for the cost, hence

$$\min_v \{J(v)\} = J(\hat{v})$$

iff the following optimality system is verified

$$\hat{z} \in H : a(\hat{z}, \varphi) = (L(\hat{v}), \varphi) + (f, \varphi) \quad \forall \varphi \in H, \quad (4.4)$$

$$\hat{p} \in H : a(\hat{p}, \phi) = (z_d - \hat{z}, \phi) \quad \forall \phi \in H, \quad (4.5)$$

$$(v_d - \hat{v}, v)_U = (L'(\hat{v}) \cdot v, \hat{p}) \quad \forall v \in U. \quad (4.6)$$

In the linear case we have $L'(\hat{v}) \cdot v = L(v)$. The optimality system is derived using the Lagrangian formalism,

$$\mathcal{L}(v, z, \phi) = \frac{1}{2} \|z - z^d\|_{L^2(\Omega)}^2 + \frac{1}{2} \|v - v^d\|_{L^2(\Gamma)}^2 + a(z, \phi) - (L(v), \phi) - (f, \phi)$$

Then the adjoint state $p \in H$ is introduced

$$\partial_z \mathcal{L}(v, z, p)(\phi) = (z - z_d, \phi)_\Omega + a(p, \phi) = 0, \quad \forall \phi \in H$$

and the gradient of the cost is obtained

$$dI(v; \eta) = \partial_v \mathcal{L}(v, z, p)(\eta) = (v - v^d, \eta)_\Gamma - (p, \eta)_\Gamma$$

which leads to the optimality condition

$$v - v^d = p \text{ a.e. on } \Gamma.$$

In the case of distributed control, we have $\Gamma = \Omega$.

Proposition *There exists the unique solution $(\hat{v}, \hat{z}, \hat{p})$ to the optimality system (4.4)-(4.6). The optimal value of the cost $J(\hat{v}) := J(\hat{v}(\Omega))$ is defined as a shape functional over the set S_{ad} . Therefore, we consider the optimum design of the network*

$$\inf_{\Omega \in S_{ad}} J(\hat{v}(\Omega)). \quad (4.7)$$

The analysis of such a variational problem requires:

1. The existence of solutions;
2. The necessary optimality conditions;
3. Finally, numerical methods for solution.

In particular, we perform the shape calculus and determine the shape gradient of the optimal control cost

$$\Omega \longmapsto J(\hat{v}(\Omega))$$

as well as the topological derivative obtained at $\epsilon := 0^+$ for the mapping

$$\epsilon \longmapsto J(\hat{v}(\Omega_\epsilon)).$$

Remark 3. *It is useful for applications to introduce a random right-hand side $f := f(\omega; x)$ to the state equation (4.1).*

4.3 Network Model

In this section, we extend the beam model previously defined on a single edge to a network structure. The model for 3D Timoshenko beams on the network takes the following form:

$$\text{Eq. (2.10)}, \quad (0, L_i) \times (0, T), i = 1 : n_e \quad (4.8a)$$

$$u_d(v_D) = 0, \quad v_D \in V_D, d \in \mathcal{E}_D, t \in (0, T) \quad (4.8b)$$

$$u_i(v_J) = u_j(v_J), \quad \forall i, j \in \mathcal{E}_J, v_J \in V_M, t \in (0, T) \quad (4.8c)$$

$$\sum_{i \in \mathcal{E}_J} \epsilon_{iJ} F_i(v_J) = 0 \quad v_J \in V_M \cup V_N, t \in (0, T) \quad (4.8d)$$

$$\sum_{i \in \mathcal{E}_J} \epsilon_{iJ} M_i(v_J) = 0 \quad v_J \in V_M \cup V_N, t \in (0, T) \quad (4.8e)$$

$$u_i(x, 0) = u_i^0 \quad x \in (0, l_i), i = 1 : n_e \quad (4.8f)$$

$$\partial_t u_i(x, 0) = u_i^1 \quad x \in (0, l_i), i = 1 : n_e \quad (4.8g)$$

The first equation describes the evolution of the system, the third, fourth, and fifth ones the continuity and balance of Kirchhoff conditions at inner nodes of the network, the remaining equations constitute the initial and boundary conditions. This system is well-posed in the space setting

$$\begin{aligned} \partial_t u, u^1 &\in \prod_{i=1}^{n_e} L^2(0, L_i); \\ u, u^0 &\in \left\{ u \in \prod_{i=1}^{n_e} L^2(0, L_i) \mid u \text{ satisfies (4.8b), (4.8c)} \right\} \end{aligned} \quad (4.9)$$

In conclusion, the model outlined here provides a comprehensive framework for understanding the behavior of 3D Timoshenko beams on networks. By incorporating dynamical equations, boundary conditions, and continuity constraints, we create a robust foundation for further analysis and optimization of network structures.

4.4 Domain Decomposition Technique for Topological Derivatives

The domain decomposition technique was used for derivation of topological derivatives for contact problems in linear elasticity [79].

The domain decomposition technique is applied in the mathematical modeling of distributed parameter systems that are defined on graphs. In this approach, a subdomain is employed to isolate the topology change from the primary section of the original domain (refer to Fig. 4.1). A Dirichlet to Neumann operator operates on the shared boundary

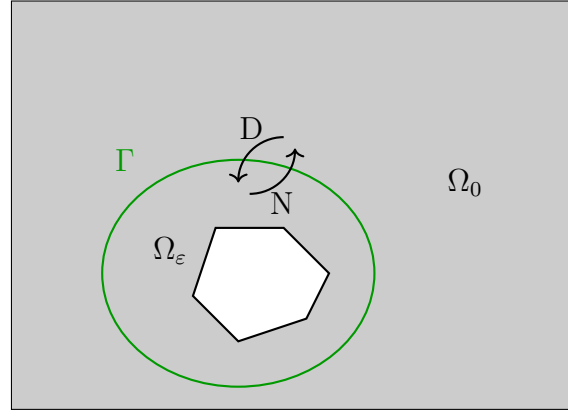


Figure 4.1: Domain Decomposition, concept[79].

of the respective domains [33]. Additionally, the domain decomposition method has been incorporated for optimizing the topology of networks, as illustrated in [43].

The graph G is decomposed into two subgraphs G_0 and G_ε (See Fig. 4.2), where the topology variations of the network are governed by the perturbations of G_ε for $\varepsilon \rightarrow 0$, in another word, a small cycle is introduced at some interior node P_0 of the network (Fig. 4.3). We assume that the vertex Q_i is a boundary vertex and the vertex P_i is the interior vertex for the subgraph G_ε which contains a small cycle.

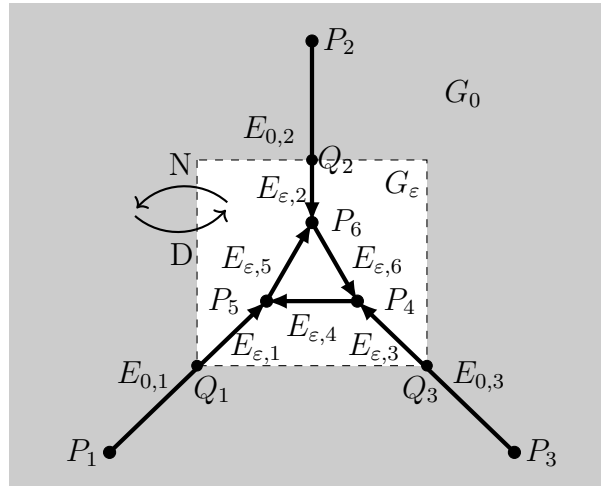


Figure 4.2: Tripod directed network with a cycle for the Timoshenko beam.

Domain decomposition technique allows us to replace the singular domain perturbation of Ω by a regular perturbation of the bilinear form for the elliptic problem considered on Ω_0 . The perturbation of bilinear form is given by the *Dirichlet-to-Neumann operator* which is also called the *Steklov-Poincaré operator*. In the case considered the nonlocal operator is represented by a matrix which can be determined explicitly. In this way, the evaluation of topological derivatives for the shape functional becomes simple and does not

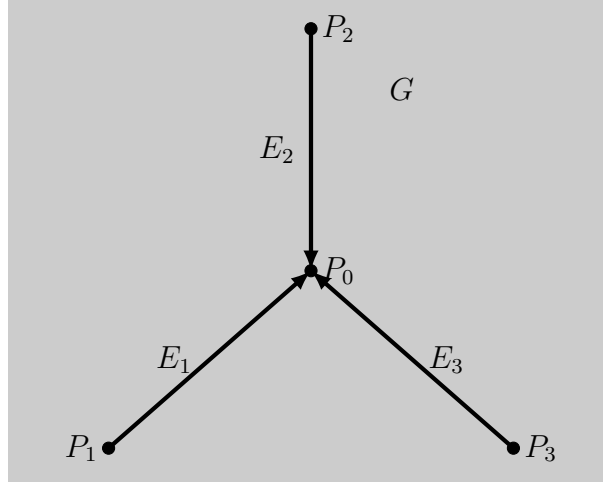


Figure 4.3: The three-star graph.

require the compound asymptotics method. The domain decomposition technique and the Steklov-Poincaré nonlocal boundary operators are used in the topological sensitivity analysis of nonlinear variational problems. Roughly speaking, Steklov-Poincaré operator for G_ε means the Dirichlet-to-Neumann operator for $G \setminus G_\varepsilon$, is a definition by Steklov-Poincaré operator for G_ε . It gives the energy of exterior domain.

4.5 Steklov-Poincaré Operator for a Subgraph G_ε with a cycle

We are going to use the domain decomposition method for a graph $G = G_0 \cup G_\varepsilon$. For example, we have $G_0 = \{E_0, V_0\}$ and $G_\varepsilon = \{E_\varepsilon, V_\varepsilon\}$ in Fig. 4.2. Here, $E_0 = \{P_1Q_1, P_2Q_2, P_3Q_3\}$ and $E_\varepsilon = \{Q_1P_5, Q_2P_6, Q_3P_4, P_4P_5, P_5P_6, P_6P_4\}$. The edges in E_0 are defined by lengths $L_{0,i} := |E_{0,i}| = L_i - \varepsilon_{\max}$ ($i = 1, 2, 3$), where L_i ($i = 1, 2, 3$) is the length of E_i in Fig. 4.3, while the edges in E_ε are characterized by lengths $|E_{\varepsilon,i}| = \varepsilon_{\max} - \varepsilon$ ($i = 1, 2, 3$), and $|E_{\varepsilon,i}| = \varepsilon$ ($i = 4, 5, 6$). It's crucial to clarify that $\varepsilon_{\max}$ represents the maximum value of ε , serving as a constant offset in E_0 . Meanwhile, ε denotes the perturbation magnitude in E_ε , distinguishing the two states. In this way the dependence of the solutions to the state equation with respect to the small parameter $\varepsilon \rightarrow 0$ is explicitly given in the weak form of the state equation.

First, we give the weak form for the full model in one beam. We introduce the test functions ψ_i , which are given by $\psi_i = (\phi_{x,i}, \phi_{y,i}, \phi_{z,i}, \eta_{x,i}, \eta_{y,i}, \eta_{z,i})^\top$, where i is the beam index. The bilinear form for a single beam can be obtained by integrating the strong system (2.12) by parts. All functions in the weak form are considered to be in the space

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$H^1(0, L_i)$. Thus, the resulting equation for a single beam is given by:

$$\begin{aligned} a(u_i, \psi_i) = & \int_0^{L_i} k^x r'_{x,i} \phi'_{x,i} + \int_0^{L_i} c^x \varphi'_{x,i} \eta'_{x,i} + \int_0^{L_i} k^s (r'_{y,i} - \varphi_{z,i}) (\phi'_{y,i} - \eta_{z,i}) \\ & + \int_0^{L_i} k^s (r'_{z,i} + \varphi_{y,i}) (\phi'_{z,i} + \eta_{y,i}) + \int_0^{L_i} c^z \varphi'_{z,i} \eta'_{z,i} + \int_0^{L_i} c^y \varphi'_{y,i} \eta'_{y,i}. \end{aligned}$$

For the distributed loading on the right-hand side, it follows the linear form

$$(q_i, \psi_i) = \int_0^{L_i} f_i^x \phi_{x,i} + \int_0^{L_i} m_i^x \eta_{x,i} + \int_0^{L_i} f_i^y \phi_{y,i} + \int_0^{L_i} m_i^y \eta_{y,i} + \int_0^{L_i} f_i^z \phi_{z,i} + \int_0^{L_i} m_i^z \eta_{z,i}.$$

Hence, the weak form of a beam is given by the equation: find $u_i \in H^1(0, L_i)$, such that

$$a(u_i, \psi_i) = (q_i, \psi_i), \quad \forall \psi \in H^1(0, L_i). \quad (4.10)$$

Thus, on the graph $G = \{E, V\}$ the total bilinear form is defined,

$$a(u, \psi) := a(\Omega; u, \psi) = \sum_{i=1}^{i=6} a_i(E_i; u_i, \psi_i).$$

We are able to determine explicitly the solution $u = u(a)$ of the Dirichlet problem $u_i(Q_i) = a_i$ for $i = 1, 2, 3$, on Ω_ε with the small cycle for $a \in \mathbb{R}^3$. The Steklov-Poincaré operator Λ_ε on G_ε is defined by the equality

$$a_\varepsilon(u(a), u(a)) = \Lambda_\varepsilon a \cdot a.$$

From the symmetry of bilinear form $a_\varepsilon(u(a), u(b)) = a_\varepsilon(u(b), u(a))$ for all $a, b \in \mathbb{R}^3$ it follows,

Proposition 1. *The matrix Λ_ε is only negative semidefinite because the energy vanishes for constant solutions of the Dirichlet problem.*

Let us return to Green's formula for the bilinear form associated with the system (4.10). We have

$$a_i(u_i, \psi_i) = \int_0^{L_i} A_i u_i \cdot \psi_i + (N_i u_i \cdot \psi_i)|_{x=0}^{x=L_i}, \quad (4.11)$$

where $u_i \mapsto N_i u_i$ is the Neumann operator N_i on $E_{0,i}$. $x = 0$ and $x = L_i$ means the value of variables for $u_i(x)$ and $\psi_i(x)$, the same as the following.

Proposition 2. *The Neumann operator N_i is given by the following vector*

Proof. We have the expressions

$$\int_0^{L_i} k_i^x r'_{x,i} \phi'_{x,i} = - \int_0^{L_i} k_i^x r''_{x,i}(x) \phi_{x,i} + (k_i^x r'_{x,i} \phi_{x,i})|_{x=0}^{x=L_i}$$

$$\begin{aligned}
\int_0^{L_i} k_i^s (r'_{y,i} - \varphi_{z,i}) \phi'_{y,i} &= - \int_0^{L_i} k_i^s (r''_{y,i} - \varphi'_{z,i}) \phi_{y,i} + (k_i^s (r'_{y,i} - \varphi_{z,i}) \phi_{y,i}) \Big|_{x=0}^{x=L_i} \\
\int_0^{L_i} k_i^s (r'_{z,i} + \varphi_{y,i}) \phi'_{z,i} &= - \int_0^{L_i} k_i^s (r''_{z,i} + \varphi'_{y,i}) \phi_{z,i} + (k_i^s (r'_{z,i} + \varphi_{y,i}) \phi_{z,i}) \Big|_{x=0}^{x=L_i} \\
\int_0^{L_i} c_i^x \varphi'_{x,i} \eta'_{x,i} &= - \int_0^{L_i} c_i^x \varphi''_{x,i}(x) \eta_{x,i}(x) + (c_i^x \varphi'_{x,i} \eta_{x,i}) \Big|_{x=0}^{x=L_i} \\
\int_0^{L_i} c_i^z \varphi'_{z,i} \eta'_{z,i} &= - \int_0^{L_i} c_i^z \varphi''_{z,i}(x) \eta_{z,i} + (c_i^z \varphi'_{z,i} \eta_{z,i}) \Big|_{x=0}^{x=L_i} \\
\int_0^{L_i} c_i^y \varphi'_{y,i} \eta'_{y,i} &= - \int_0^{L_i} c_i^y \varphi''_{y,i}(x) \eta_{y,i} + (c_i^y \varphi'_{y,i} \eta_{y,i}) \Big|_{x=0}^{x=L_i}
\end{aligned}$$

Thus we find six components of $N_i u_i$ given by

$$\begin{aligned}
(N_i u_i \cdot \psi_i) \Big|_{x=0}^{x=L_i} &= (k_i^x r'_{x,i} \phi_{x,i}) \Big|_{x=0}^{x=L_i} + (k_i^s (r'_{y,i} - \varphi_{z,i}) \phi_{y,i}) \Big|_{x=0}^{x=L_i} + (k_i^s (r'_{z,i} + \varphi_{y,i}) \phi_{z,i}) \Big|_{x=0}^{x=L_i} \\
&\quad + (c_i^x \varphi'_{x,i} \eta_{x,i}) \Big|_{x=0}^{x=L_i} + (c_i^y \varphi'_{y,i} \eta_{y,i}) \Big|_{x=0}^{x=L_i} + (c_i^z \varphi'_{z,i} \eta_{z,i}) \Big|_{x=0}^{x=L_i}
\end{aligned}$$

This leads to the result on the Steklov-Poincaré operator, we should use the exact solution from Theorem 2.2 in the Ogiermann Dissertation, as well as replace the traces of test functions at $x = 0$ by the Dirichlet conditions. $\square$

Proposition 3. *If we know the exact solution u_a for the Dirichlet problem on the graph G_ε with the polynomials on the edges $\text{col}(r_x, r_y, r_z, \varphi_x, \varphi_y, \varphi_z)$, it follows that the associated energy for such a solution takes the form $a(u_a, u_a) = a^\top \Lambda_\varepsilon \cdot a$, thus the energy functional for the graph G reads*

$$\psi \mapsto a(\Omega; \psi, \psi) = a(\Omega_0; \psi, \psi) - \psi(L - \varepsilon_{\max})^\top \Lambda_\varepsilon \cdot \psi(L - \varepsilon_{\max}), \quad (4.12)$$

where $0 < \varepsilon \leq \varepsilon_{\max}$. The restriction of the state equation to Ω_0 is considered under the assumption that the control is supported in Ω_0 and the state is observed in Ω_0 . Thus, the state equation becomes: Find $u \in H(\Omega_0)$ such that

$$a(\Omega_0; u, \psi) - u(L - \varepsilon_{\max})^\top \Lambda(\varepsilon) \cdot \psi(L - \varepsilon_{\max}) = (q, \psi)_{\Omega_0}, \quad \forall \psi \in H(\Omega_0). \quad (4.13)$$

In this way, the dependence of the state equation on the small cycle in G is replaced by the dependence of Λ_ε on the cycle.

Proposition 4. *The Green's formula for the state equation in truncated domain Ω_0 reads*

$$a(\Omega_0; u, \psi) = \sum_{i=1}^{i=6} a_i(u_i, \psi_i) = - \sum_{i=1}^{i=6} (A_i u_i, \psi_i)_{E_i} - \sum_{i=1}^{i=3} N_i u_i(0) \cdot \psi(0). \quad (4.14)$$

For the new state equation, the optimality system is defined in the same way as before, on the domain Ω_0 . The topological derivative of the optimal cost is defined as the limit of derivatives for $\varepsilon \rightarrow 0^+$.

4.6 Explicit Solution to the Static Timoshenko Beam Model

4.6.1 Solution to the 1D Timoshenko Beam Model

Then we derive the solution to the static model. Assume $f_i(x) = m_i(x) = 0$. We will show the explicit solution for both reduced and full models. First, it's the solution for the reduced model.

Proposition 5. *Assume that $f_i^z(x) = 0$ and $m_i^y(x) = 0$. Let us consider the system*

$$\begin{aligned} f_i^z(x) + k_i^s (r_{z,i}''(x) + \varphi_{y,i}'(x)) &:= f_i^z(x) + k_i^s (u_1''(x) + u_2'(x)) = 0, \\ m_i^y(x) + c_i^y \varphi_{y,i}''(x) - k_i^s (r_{z,i}'(x) + \varphi_{y,i}(x)) &:= m_i^y(x) + c_i^y u_2''(x) - k_i^s (u_1'(x) + u_2(x)) = 0, \end{aligned}$$

of two linear differential equations for $u(x) = (u_1(x), u_2(x))^T$ with the Dirichlet boundary conditions $u(0) = (u_1(0), u_2(0))^T$ and $u(L_i) = (u_1(L_i), u_2(L_i))^T$. The solution is given by

$$\begin{aligned} u_1(x) &= \frac{\lambda_1}{k_i^s} x - \frac{1}{6c_i^y} \lambda_1 x^3 - \frac{1}{2c_i^y} \lambda_2 x^2 - \frac{1}{c_i^y} \lambda_3 x + \lambda_4, \\ u_2(x) &= \frac{1}{2c_i^y} \lambda_1 x^2 + \frac{1}{c_i^y} \lambda_2 x + \frac{1}{c_i^y} \lambda_3, \end{aligned} \tag{4.15}$$

where $\lambda_i, i = 1, 2, 3, 4$, are determined in terms of the nonhomogeneous Dirichlet conditions

$$\begin{aligned} \lambda_1 &= \frac{12c_i^y}{L_i(12c_i^y k_{s,i}^{-1} + L_i^2)} (u_1(L_i) - u_1(0)) + \frac{6c_i^y}{12c_i^y k_{s,i}^{-1} + L_i^2} (u_2(L_i) + u_2(0)), \\ \lambda_2 &= \frac{12c_{i,y}^2}{L_i k (12c_i^y k_{s,i}^{-1} + L_i^2)} (u_2(L_i) - u_2(0)) \\ &\quad + \frac{2c_i^y}{12c_i^y k_{s,i}^{-1} + L_i^2} (3u_1(0) - 3u_1(L_i) - 2L_i u_2(0) - L_i u_2(L_i)), \\ \lambda_3 &= c_i^y u_2(0), \\ \lambda_4 &= u_1(0). \end{aligned} \tag{4.16}$$

Proof. Set $u(x) = (r_z, \varphi_y)^T$. The Dirichlet problem for $u(0) = (u_1(0), u_2(0))^T = (r_z(0), \varphi_y(0))^T$, $u(L_i) = (u_1(L_i), u_2(L_i))^T = (r_z(L_i), \varphi_y(L_i))^T$ are given. The equation becomes

$$\begin{aligned} k_i^s (r_{z,i}''(x) + \varphi_{y,i}'(x)) &= 0, \\ c_i^y \varphi_{y,i}''(x) - k_i^s (r_{z,i}'(x) + \varphi_{y,i}(x)) &= 0. \end{aligned}$$

Hence $k_i^s (r_{z,i}'(x) + \varphi_{y,i}(x)) = \lambda_1$ thus $c_i^y \varphi_{y,i}''(x) = \lambda_1$, therefore $c_i^y \varphi_{y,i}'(x) = \lambda_1 x + \lambda_2$ and

$c_i^y \varphi_{y,i}(x) = \frac{1}{2} \lambda_1 x^2 + \lambda_2 x + \lambda_3$. It leads to

$$\begin{aligned}\varphi_{y,i}(x) &= \frac{1}{2c_i^y} \lambda_1 x^2 + \frac{1}{c_i^y} \lambda_2 x + \frac{1}{c_i^y} \lambda_3, \\ r'_{z,i}(x) &= \frac{\lambda_1}{k_i^s} - \varphi_{y,i}(x) = \frac{\lambda_1}{k_i^s} - \frac{1}{2c_i^y} \lambda_1 x^2 - \frac{1}{c_i^y} \lambda_2 x - \frac{1}{c_i^y} \lambda_3, \\ r_{z,i}(x) &= \frac{\lambda_1}{k_i^s} x - \frac{1}{6c_i^y} \lambda_1 x^3 - \frac{1}{2c_i^y} \lambda_2 x^2 - \frac{1}{c_i^y} \lambda_3 x + \lambda_4.\end{aligned}$$

Henceforth, the constants λ_i are determined from the Dirichlet boundary conditions

$$\begin{aligned}r_{z,i}(0) &= \lambda_4 = u_1(0), \\ \varphi_{y,i}(0) &= \frac{1}{c_i^y} \lambda_3 = u_2(0), \\ r_{z,i}(L_i) &= \frac{\lambda_1}{k_i^s} L_i - \frac{1}{6c_i^y} \lambda_1 L_i^3 - \frac{1}{2c_i^y} \lambda_2 L_i^2 - \frac{1}{c_i^y} \lambda_3 L_i + \lambda_4 = u_1(L_i), \\ \varphi_{y,i}(L_i) &= \frac{1}{2c_i^y} \lambda_1 L_i^2 + \frac{1}{c_i^y} \lambda_2 L_i + \frac{1}{c_i^y} \lambda_3 = u_2(L_i).\end{aligned}$$

Thus we determine the unique solution Eq. (4.16) of the model in function of Dirichlet conditions. $\square$

Remark 4. *The passage to the limit $L_i \rightarrow 0$ in the solution given by Proposition 5 can be performed if the singular terms vanish, i.e., $u_1(L_i) - u_1(0) = 0$ and $u_2(L_i) - u_2(0) = 0$, otherwise boundary layers appear.*

Now, we give the matrix notation for exact solutions of Dirichlet boundary value problems. We define the matrix functions $Q_i, R_i : [0, L_i] \rightarrow \mathbb{R}^{2 \times 2}$ by

$$Q_i(x) := \begin{pmatrix} a_{11}^{i,0}(x) & a_{12}^{i,0}(x) \\ a_{21}^{i,0}(x) & a_{22}^{i,0}(x) \end{pmatrix}$$

with the coefficient functions

$$\begin{aligned}a_{11}^{i,0}(x) &:= \mu_i^y (L_i - x) (L_i^2 + L_i x - 2x^2 + 12k_{s,i}^{-1} c_i^y), \\ a_{22}^{i,0}(x) &:= \mu_i^y (L_i - x) (L_i^2 - 3L_i x + 12k_{s,i}^{-1} c_i^y), \\ a_{12}^{i,0}(x) &:= -\mu_i^y x (L_i - x) (L_i^2 - L_i x + 6k_{s,i}^{-1} c_i^y), \\ a_{21}^{i,0}(x) &:= 6\mu_i^y x (L_i - x),\end{aligned}$$

where

$$\mu_i^y := (L_i^3 + 12k_{s,i}^{-1} c_i^y L_i)^{-1} = \frac{k_i^s}{L_i(k_i^s L_i^2 + 12c_i^y)},$$

and the singular term at $L_i = 0$. Furthermore

$$R_i(x) := \begin{pmatrix} a_{11}^{i,L}(x) & a_{12}^{i,L}(x) \\ a_{21}^{i,L}(x) & a_{22}^{i,L}(x) \end{pmatrix} \quad (4.17)$$

with coefficient functions

$$\begin{aligned} a_{11}^{i,L}(x) &:= \mu_i^y x (3L_i x - 2x^2 + 12k_{s,i}^{-1} c_i^y), \\ a_{22}^{i,L}(x) &:= \mu_i^y x (-2L_i^2 + 3L_i x + 12k_{s,i}^{-1} c_i^y), \\ a_{12}^{i,L}(x) &:= \mu_i^y x (L_i - x) (L_i x + 6k_{s,i}^{-1} c_i^y), \\ a_{21}^{i,L}(x) &:= -6\mu_i^y x (L_i - x). \end{aligned}$$

Then, the exact solution of the Dirichlet problem for a single beam takes the form

$$u_i(x) = Q_i(x)u_i(0) + R_i(x)u_i(L_i). \quad (4.18)$$

4.6.2 Solution to the 3D Timoshenko Beam Model

For the 3D full model, we refer to the explicit solution presented in [61]. These findings will be unified into a single six-dimensional vector, which will inherently depend on the boundary conditions at the global coordinate level. Here, we define the matrix functions $Q_i, R_i : [0, L_i] \rightarrow \mathbb{R}^{6 \times 6}$ by

$$Q_i(x) := \begin{pmatrix} \mathbb{E}_i & \mathbb{O}_3 \\ \mathbb{O}_3 & \mathbb{E}_i \end{pmatrix} \begin{pmatrix} a_{11}^{i,0}(x) & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22}^{i,0}(x) & 0 & 0 & 0 & a_{26}^{i,0}(x) \\ 0 & 0 & a_{33}^{i,0}(x) & 0 & a_{35}^{i,0}(x) & 0 \\ 0 & 0 & 0 & a_{44}^{i,0}(x) & 0 & 0 \\ 0 & 0 & a_{53}^{i,0}(x) & 0 & a_{55}^{i,0}(x) & 0 \\ 0 & a_{62}^{i,0}(x) & 0 & 0 & 0 & a_{66}^{i,0}(x) \end{pmatrix} \begin{pmatrix} \mathbb{E}_i^\top & \mathbb{O}_3 \\ \mathbb{O}_3 & \mathbb{E}_i^\top \end{pmatrix} \quad (4.19)$$

with the coefficient functions

$$\begin{aligned} a_{11}^{i,0}(x) &:= L_i^{-1} (L_i - x), \\ a_{22}^{i,0}(x) &:= \mu_i^z (L_i - x) (L_i^2 + L_i x - 2x^2 + 12k_{s,i}^{-1} c_i^z), \\ a_{33}^{i,0}(x) &:= \mu_i^y (L_i - x) (L_i^2 + L_i x - 2x^2 + 12k_{s,i}^{-1} c_i^y), \\ a_{44}^{i,0}(x) &:= L_i^{-1} (L_i - x), \\ a_{55}^{i,0}(x) &:= \mu_i^y (L_i - x) (L_i^2 - 3L_i x + 12k_{s,i}^{-1} c_i^y), \\ a_{66}^{i,0}(x) &:= \mu_i^z (L_i - x) (L_i^2 - 3L_i x + 12k_{s,i}^{-1} c_i^z), \\ a_{26}^{i,0}(x) &:= \mu_i^z x (L_i - x) (L_i^2 - L_i x + 6k_{s,i}^{-1} c_i^z), \\ a_{35}^{i,0}(x) &:= -\mu_i^y x (L_i - x) (L_i^2 - L_i x + 6k_{s,i}^{-1} c_i^y), \end{aligned}$$

$$\begin{aligned} a_{53}^{i,0}(x) &:= 6\mu_i^y x (L_i - x), \\ a_{62}^{i,0}(x) &:= -6\mu_i^z x (L_i - x), \end{aligned}$$

where

$$\mu_i^z := (L_i^3 + 12k_{s,i}^{-1}c_i^z L_i)^{-1} = \frac{k_i^s}{L_i(k_i^s L_i^2 + 12c_i^z)},$$

as well as

$$R_i(x) := \begin{pmatrix} \mathbb{E}_i & \mathbb{O}_3 \\ \mathbb{O}_3 & \mathbb{E}_i \end{pmatrix} \begin{pmatrix} a_{11}^{i,L}(x) & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22}^{i,L}(x) & 0 & 0 & 0 & a_{26}^{i,L}(x) \\ 0 & 0 & a_{33}^{i,L}(x) & 0 & a_{35}^{i,L}(x) & 0 \\ 0 & 0 & 0 & a_{44}^{i,L}(x) & 0 & 0 \\ 0 & 0 & a_{53}^{i,L}(x) & 0 & a_{55}^{i,L}(x) & 0 \\ 0 & a_{62}^{i,L}(x) & 0 & 0 & 0 & a_{66}^{i,L}(x) \end{pmatrix} \begin{pmatrix} \mathbb{E}_i^\top & \mathbb{O}_3 \\ \mathbb{O}_3 & \mathbb{E}_i^\top \end{pmatrix}$$

with the coefficient functions

$$\begin{aligned} a_{11}^{i,L}(x) &:= L_i^{-1}x, \\ a_{22}^{i,L}(x) &:= \mu_i^z x (3L_i x - 2x^2 + 12k_{s,i}^{-1}c_i^z), \\ a_{33}^{i,L}(x) &:= \mu_i^y x (3L_i x - 2x^2 + 12k_{s,i}^{-1}c_i^y), \\ a_{44}^{i,L}(x) &:= L_i^{-1}x, \\ a_{55}^{i,L}(x) &:= \mu_i^y x (-2L_i^2 + 3L_i x + 12k_{s,i}^{-1}c_i^y), \\ a_{66}^{i,L}(x) &:= \mu_i^z x (-2L_i^2 + 3L_i x + 12k_{s,i}^{-1}c_i^z), \\ a_{26}^{i,L}(x) &:= \mu_i^z x (L_i - x) (L_i x + 6k_{s,i}^{-1}c_i^z), \\ a_{35}^{i,L}(x) &:= -\mu_i^y x (L_i - x) (L_i x + 6k_{s,i}^{-1}c_i^y), \\ a_{53}^{i,L}(x) &:= -6\mu_i^y x (L_i - x), \\ a_{62}^{i,L}(x) &:= 6\mu_i^z x (L_i - x). \end{aligned}$$

Similarly to the reduced model, the exact solution for the homogeneous full model could also be written as

$$u_i(x) = Q_i(x)u_i(0) + R_i(x)u_i(L_i). \quad (4.20)$$

4.7 Dirichlet-to-Neumann Operator subgraphs for Timoshenko Beams Networks

We start with the single beam. For the reduced model, we introduce the notation for matrix functions depending on the length L_i of the beam

$$\begin{aligned}
D_i^I &:= \begin{pmatrix} -12c_i^y \mu_i^y & 6c_i^y \mu_i^y L_i \\ 6c_i^y \mu_i^y L_i & -(L_i^{-1} + 3\mu_i^y L_i^2) c_i^y \end{pmatrix}, \\
D_i^{II} &:= \begin{pmatrix} 12c_i^y \mu_i^y & 6c_i^y \mu_i^y L_i \\ -6c_i^y \mu_i^y L_i & (L_i^{-1} - 3\mu_i^y L_i^2) c_i^y \end{pmatrix}, \\
D_i^{III} &:= \begin{pmatrix} -12c_i^y \mu_i^y & 6c_i^y \mu_i^y L_i \\ -6c_i^y \mu_i^y L_i & (3\mu_i^y L_i^2 - L_i^{-1}) c_i^y \end{pmatrix}, \\
D_i^{IV} &:= \begin{pmatrix} 12c_i^y \mu_i^y & 6c_i^y \mu_i^y L_i \\ 6c_i^y \mu_i^y L_i & (L_i^{-1} + 3\mu_i^y L_i^2) c_i^y \end{pmatrix}.
\end{aligned} \tag{4.21}$$

Then, for the full model, we define the matrices

$$\begin{aligned}
D_i^F &:= \mathbb{E}_i \begin{pmatrix} L_i^{-1} k_i^x & 0 & 0 \\ 0 & 12c_i^z \mu_i^z & 0 \\ 0 & 0 & 12c_i^y \mu_i^y \end{pmatrix} \mathbb{E}_i^\top, \\
D_i^+ &:= \mathbb{E}_i \begin{pmatrix} L_i^{-1} c_i^x & 0 & 0 \\ 0 & c_i^y (L_i^{-1} + 3\mu_i^y L_i^2) & 0 \\ 0 & 0 & c_i^z (L_i^{-1} + 3\mu_i^z L_i^2) \end{pmatrix} \mathbb{E}_i^\top, \\
D_i^- &:= \mathbb{E}_i \begin{pmatrix} L_i^{-1} c_i^x & 0 & 0 \\ 0 & c_i^y (L_i^{-1} + 3\mu_i^y L_i^2) & 0 \\ 0 & 0 & c_i^z (L_i^{-1} - 3\mu_i^z L_i^2) \end{pmatrix} \mathbb{E}_i^\top, \\
S_i &:= \mathbb{E}_i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 6c_i^y \mu_i^y L_i \\ 0 & -6c_i^z \mu_i^z L_i & 0 \end{pmatrix} \mathbb{E}_i^\top,
\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}
D_i^I &= - \begin{pmatrix} D_i^F & -S_i^\top \\ -S_i & D_i^+ \end{pmatrix}, D_i^{II} := \begin{pmatrix} D_i^F & S_i^\top \\ -S_i & D_i^- \end{pmatrix}, \\
D_i^{III} &:= - \begin{pmatrix} D_i^F & -S_i^\top \\ S_i & D_i^- \end{pmatrix}, D_i^{IV} := \begin{pmatrix} D_i^F & S_i^\top \\ S_i & D_i^+ \end{pmatrix}.
\end{aligned} \tag{4.23}$$

We use the same notation for both models. With this notation, we could obtain the following representation of the *Dirichlet-to-Neumann* (DtN) mapping for the single beam, i.e., the Dirichlet boundary conditions $u_i(0), u_i(L_i)$ to the vectors $F_i(0), M_i(0), F_i(L_i)$ and $M_i(L_i)$.

Theorem 9. *The Dirichlet-to-Neumann operator of a single beam with zero distributed*

loads $f_i, m_i : [0, L_i] \rightarrow \mathbb{R}^d (d = 1, 3)$ is given by

$$\begin{aligned} \begin{pmatrix} F_i(0) \\ M_i(0) \end{pmatrix} &= D_i^I u_i(0) + D_i^{II} u_i(L_i), \\ \begin{pmatrix} F_i(L_i) \\ M_i(L_i) \end{pmatrix} &= D_i^{III} u_i(0) + D_i^{IV} u_i(L_i), \end{aligned} \quad (4.24)$$

where $D_i^I, D_i^{II}, D_i^{III}, D_i^{IV}$ are (4.21) and (4.23) for 1D and 3D, respectively. And $f_i(x) = f_i^z(x), m_i(x) = m_i^z(x), F_i(x) = F_i^z(x), M_i(x) = M_i^y(x)$ for reduced model.

Then the elastic energy for the beam is evaluated

$$\begin{aligned} \mathcal{E}(u_i(0), u_i(L_i)) &= -u_i(0)^\top \begin{pmatrix} F_i(0) \\ M_i(0) \end{pmatrix} - u_i(L_i)^\top \begin{pmatrix} F_i(L_i) \\ M_i(L_i) \end{pmatrix} \\ &= -u_i(0)^\top D_i^I u_i(0) - u_i(0)^\top D_i^{II} u_i(L_i) \\ &\quad - u_i(L_i)^\top D_i^{III} u_i(0) - u_i(L_i)^\top D_i^{IV} u_i(L_i). \end{aligned} \quad (4.25)$$

There is the negative semidefinite matrix Λ such that

$$\mathcal{E}(u_i(0), u_i(L_i)) = \begin{pmatrix} u_i(0) \\ u_i(L_i) \end{pmatrix}^\top \Lambda \begin{pmatrix} u_i(0) \\ u_i(L_i) \end{pmatrix}. \quad (4.26)$$

The Steklov-Poincaré operator for the Timoshenko beam maps the Dirichlet boundary conditions $u_i(0), u_i(L_i)$ to the vectors $F_i(0), M_i(0), F_i(L_i)$ and $M_i(L_i)$, which are the Neumann boundary conditions for the remaining part of the network connected to the specific beam.

4.8 Steklov-Poincaré Operator for the Cross with Small Cycle

We obtain the exact solution for the 1D and 3D Timoshenko beam model on the cross with the cycle. In this way, the Steklov-Poincaré operator is identified for the purposes of domain decomposition technique for modeling and design of the network.

This leads to the matrix representation of the Steklov-Poincaré operator for the cross $G = \{E, V\}$ with the small cycle $G_\varepsilon = \{E_\varepsilon, V_\varepsilon\}$ at the center, of the size $\varepsilon \rightarrow 0$. Such a system is called a network and its model includes a graph $G = \{V, E\}$ along with the state equations on edges E . Such a model is used e.g., in analysis of real life large scale gas networks with quasilinear hyperbolic equations on the edges. At the interior nodes of the graph there are prescribed transmission conditions, the continuity of solutions as well as the continuity of fluxes. There is no external force at G_ε . So the transmission conditions at the joint verticals on G_ε becomes

$$\begin{pmatrix} F_1(\varepsilon_{\max} - \varepsilon) \\ M_1(\varepsilon_{\max} - \varepsilon) \end{pmatrix} + \begin{pmatrix} F_4(\varepsilon) \\ M_4(\varepsilon) \end{pmatrix} - \begin{pmatrix} F_5(0) \\ M_5(0) \end{pmatrix} = 0, \quad u_1(\varepsilon_{\max} - \varepsilon) = u_4(\varepsilon) = u_5(0), \quad (4.27a)$$

$$\begin{pmatrix} F_2(\varepsilon_{\max} - \varepsilon) \\ M_2(\varepsilon_{\max} - \varepsilon) \end{pmatrix} + \begin{pmatrix} F_5(\varepsilon) \\ M_5(\varepsilon) \end{pmatrix} - \begin{pmatrix} F_6(0) \\ M_6(0) \end{pmatrix} = 0, \quad u_2(\varepsilon_{\max} - \varepsilon) = u_5(\varepsilon) = u_6(0), \quad (4.27b)$$

$$\begin{pmatrix} F_3(\varepsilon_{\max} - \varepsilon) \\ M_3(\varepsilon_{\max} - \varepsilon) \end{pmatrix} + \begin{pmatrix} F_6(\varepsilon) \\ M_6(\varepsilon) \end{pmatrix} - \begin{pmatrix} F_4(0) \\ M_4(0) \end{pmatrix} = 0, \quad u_3(\varepsilon_{\max} - \varepsilon) = u_6(\varepsilon) = u_4(0). \quad (4.27c)$$

Here, F_i, M_i, u_i ($i = 1, 2, 3$) is the defined on the edge $E_{\varepsilon,i}$. Using the formula Eq. (4.24), where the length of edge is $\varepsilon_{\max} - \varepsilon$, the first balanced equation in (4.27a) can be written as

$$D_1^{\text{III}}u_1(0) + D_1^{\text{IV}}u_1(\varepsilon_{\max} - \varepsilon) + D_4^{\text{III}}u_4(0) + D_4^{\text{IV}}u_4(\varepsilon) - D_5^{\text{I}}u_5(0) - D_5^{\text{II}}u_5(\varepsilon) = 0.$$

By the continuity, we have

$$\begin{aligned} D_1^{\text{III}}u_1(0) + D_1^{\text{IV}}u_1(\varepsilon_{\max} - \varepsilon) + D_4^{\text{III}}u_3(\varepsilon_{\max} - \varepsilon) \\ + D_4^{\text{IV}}u_1(\varepsilon_{\max} - \varepsilon) - D_5^{\text{I}}u_1(\varepsilon_{\max} - \varepsilon) - D_5^{\text{II}}u_2(\varepsilon_{\max} - \varepsilon) = 0, \end{aligned}$$

namely,

$$(D_1^{\text{IV}} + D_4^{\text{IV}} - D_5^{\text{I}})u_1(\varepsilon_{\max} - \varepsilon) - D_5^{\text{II}}u_2(\varepsilon_{\max} - \varepsilon) + D_4^{\text{III}}u_3(\varepsilon_{\max} - \varepsilon) = -D_1^{\text{III}}u_1(0). \quad (4.28)$$

Similarly, from Eq. (4.27b) and Eq. (4.27c), we get the following equations,

$$D_5^{\text{III}}u_1(\varepsilon_{\max} - \varepsilon) + (D_2^{\text{IV}} + D_5^{\text{IV}} - D_6^{\text{I}})u_2(\varepsilon_{\max} - \varepsilon) - D_6^{\text{II}}u_3(\varepsilon_{\max} - \varepsilon) = -D_2^{\text{III}}u_2(0), \quad (4.29)$$

$$-D_4^{\text{II}}u_1(\varepsilon_{\max} - \varepsilon) + D_6^{\text{III}}u_2(\varepsilon_{\max} - \varepsilon) + (D_3^{\text{IV}} + D_6^{\text{IV}} - D_4^{\text{I}})u_3(\varepsilon_{\max} - \varepsilon) = -D_3^{\text{III}}u_3(0). \quad (4.30)$$

Combine Eq. (4.28), Eq. (4.29) and Eq. (4.30), we have

$$\begin{aligned} \begin{pmatrix} D_1^{\text{IV}} + D_4^{\text{IV}} - D_5^{\text{I}} & -D_5^{\text{II}} & D_4^{\text{III}} \\ D_5^{\text{III}} & D_2^{\text{IV}} + D_5^{\text{IV}} - D_6^{\text{I}} & -D_6^{\text{II}} \\ -D_4^{\text{II}} & D_6^{\text{III}} & D_3^{\text{IV}} + D_6^{\text{IV}} - D_4^{\text{I}} \end{pmatrix} \begin{pmatrix} u_1(\varepsilon_{\max} - \varepsilon) \\ u_2(\varepsilon_{\max} - \varepsilon) \\ u_3(\varepsilon_{\max} - \varepsilon) \end{pmatrix} \\ = - \begin{pmatrix} D_1^{\text{III}} & 0 & 0 \\ 0 & D_2^{\text{III}} & 0 \\ 0 & 0 & D_3^{\text{III}} \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \end{pmatrix}. \end{aligned} \quad (4.31)$$

That is, we could use $u_i(0)$ to present $u_i(\varepsilon_{\max} - \varepsilon)$. The Eq.(4.24) will be

$$\begin{pmatrix} F_i(0) \\ M_i(0) \end{pmatrix} = \Lambda_\varepsilon u_i(0),$$

where

$$\begin{aligned} \Lambda_\varepsilon &= \begin{pmatrix} D_1^I & 0 & 0 \\ 0 & D_2^I & 0 \\ 0 & 0 & D_3^I \end{pmatrix} - \begin{pmatrix} D_1^{II} & 0 & 0 \\ 0 & D_2^{II} & 0 \\ 0 & 0 & D_3^{II} \end{pmatrix} \\ &\quad \begin{pmatrix} D_1^{IV} + D_4^{IV} - D_5^I & -D_5^{II} & D_4^{III} \\ D_5^{III} & D_2^{IV} + D_5^{IV} - D_6^I & -D_6^{II} \\ -D_4^{II} & D_6^{III} & D_3^{IV} + D_6^{IV} - D_4^I \end{pmatrix}^{-1} \begin{pmatrix} D_1^{III} & 0 & 0 \\ 0 & D_2^{III} & 0 \\ 0 & 0 & D_3^{III} \end{pmatrix} \\ &:= D_1 - D_2 A^{-1} D_3 \end{aligned} \tag{4.32}$$

is an operator with respect to ε .

In both 1D and 3D model, the structure of the matrix Λ_ε remains uniform. We distinctly utilize the matrix $D_i^j (i = 1, 2, \dots, 6, j = I, II, III, IV)$ to denote the corresponding cases accordingly. Obviously, in 1D, the operator Λ_ε is a 6×6 matrix while in 3D, it assumes dimensions of 18×18 .

Our goal now is to identify such a matrix $\varepsilon \rightarrow \Lambda_\varepsilon$, as well as its derivative at ε_0 for the cross with a small cycle of size $\varepsilon \rightarrow 0$. This leads us to the topological derivative of shape functionals with respect to nucleation of the cycle. The matrix Λ_ε is a representation of the so-called *Steklov-Poincaré* operator for the network at consideration.

4.9 Topological Derivative of Optimal Control Cost for Static Problem

In this section, a new result of the topological derivative for Timoshenko beams is presented.

We use the standard technique for shape [80] and topology optimization [58]. Namely, the material derivatives are employed in the shape sensitivity analysis in the framework of speed method [80].

Let us construct the prescribed state (z, v) on the graph with a small cycle that $\varepsilon = \varepsilon_0$ first. Control v_i is applied to beams $E_{0,i}$. Then we can obtain the corresponding state z_i from the network modeling. In this way, we evaluate the Topological Derivative of the Optimal Cost for the singular perturbation of the graph by a small cycle. The control problem's cost functional can be represented as follows:

$$J(\Omega) = \frac{1}{2} \sum_{i=1}^3 \int_0^{L_i - \varepsilon_{\max}} (u_i - z_i)^2 + (q_i - v_i)^2. \tag{4.33}$$

The weak form on G_0 is

$$a_0(u, \varphi) - u(L - \varepsilon_{\max})^\top \Lambda_\varepsilon \varphi(L - \varepsilon_{\max}) = (q, \varphi) \quad \forall \varphi \in V,$$

where the space V is defined as

$$V = \{ \varphi = (\varphi_1, \varphi_2, \varphi_3) \mid \varphi_i \in H^1(0, L_i), \varphi_i(P_i) = 0, i = 1, 2, 3 \}. \quad (4.34)$$

The Lagrangian is defined as

$$\begin{aligned} \mathcal{L}(u, q, p) = & \frac{1}{2} \sum_{i=1}^3 \int_0^{L_i - \varepsilon_{\max}} (u_i - z_i)^2 + (q - v)^2 \\ & + a_0(u, p) - p(L - \varepsilon_{\max})^\top \Lambda_\varepsilon u(L - \varepsilon_{\max}) - (q, p), \end{aligned} \quad (4.35)$$

where p is the adjoint variable, $p(L - \varepsilon_{\max}) = (p_1(L_1 - \varepsilon_{\max}), p_2(L_2 - \varepsilon_{\max}), p_3(L_3 - \varepsilon_{\max}))^\top$ and $u(L - \varepsilon_{\max}) = (u_1(L_1 - \varepsilon_{\max}), u_2(L_2 - \varepsilon_{\max}), u_3(L_3 - \varepsilon_{\max}))^\top$. The optimality system is derived by applying the calculus of variations to the Lagrangian and solving the resulting Euler-Lagrange equations. Specifically, the adjoint state equation is obtained by solving $\frac{\partial \mathcal{L}}{\partial u} = 0$, which yields

$$\sum_{i=1}^3 \int_0^{L_i - \varepsilon_{\max}} (u_i - z_i) \eta_i + a_0(\eta, p) - \eta(L - \varepsilon_{\max})^\top \Lambda_\varepsilon p(L - \varepsilon_{\max}) = 0, \quad (4.36)$$

i.e.,

$$(u - z, \eta)_{\Omega_0} + a_0(\eta, p) - \eta(L - \varepsilon_{\max})^\top \Lambda_\varepsilon p(L - \varepsilon_{\max}) = 0, \quad (4.37)$$

where $\eta(L - \varepsilon_{\max})$ defined is similar to $p(L - \varepsilon_{\max})$.

Similarly, solving $\frac{\partial \mathcal{L}}{\partial v} = 0$ leads to the optimal control,

$$q = p + v.$$

The optimality system is given by the following coupled equations:

$$\begin{cases} (u, \varphi)_{\Omega_0} + a_0(p, \eta) - p(L - \varepsilon_{\max})^\top \Lambda_\varepsilon \varphi(L - \varepsilon_{\max}) = (z, \varphi)_{\Omega_0} \\ a_0(u, \varphi) - (p, \varphi)_{\Omega_0} - u(L - \varepsilon_{\max})^\top \Lambda_\varepsilon \varphi(L - \varepsilon_{\max}) = (v, \varphi)_{\Omega_0} \end{cases} \quad (4.38)$$

Then we get the topological derivative of cost functional,

$$\frac{dJ}{d\varepsilon} = (u_i - z_i, \dot{u}_i)_{\Omega_0} + (q_i - v_i, \dot{q}_i)_{\Omega_0}. \quad (4.39)$$

The derivative of the system (4.38) is

$$\begin{cases} (\dot{u}, \varphi)_{\Omega_0} + a_0(\dot{p}, \eta) - \dot{p}(L - 1)^\top \Lambda_\varepsilon \varphi(L - \varepsilon_{\max}) - p(L - 1)^\top \dot{\Lambda}_\varepsilon \varphi(L - \varepsilon_{\max}) = 0, \\ a_0(\dot{u}, \varphi) - (\dot{p}, \varphi)_{\Omega_0} - \dot{u}(L - 1)^\top \Lambda_\varepsilon \varphi(L - \varepsilon_{\max}) - u(L - 1)^\top \dot{\Lambda}_\varepsilon \varphi(L - \varepsilon_{\max}) = 0. \end{cases} \quad (4.40)$$

The second order topological derivatives [68], [69], [14] can be also identified.

Since it's not easy to get the inverse of a 18×18 symbolic matrix, we could use the matrix decomposition trick. For matrix A in (4.32), we use LU decomposition, i.e., $A = \bar{L}\bar{U}$, where the $\bar{L}$ matrix is a lower triangular matrix, with all diagonal elements equal to 1 and all elements above the diagonal equal to 0. The $\bar{U}$ matrix is an upper triangular matrix, with all elements below the diagonal equal to 0. Since $A^{-1} = \bar{U}^{-1}\bar{L}^{-1}$, we have

$$\Lambda_\varepsilon = D_1 - D_2\bar{U}^{-1}\bar{L}^{-1}D_3,$$

therefore,

$$\dot{\Lambda}_\varepsilon = \frac{d\Lambda_\varepsilon}{d\varepsilon} = \frac{dD_1}{d\varepsilon} - \frac{dD_2}{d\varepsilon}\bar{U}^{-1}\bar{L}^{-1}D_3 - D_2\left(\frac{d\bar{U}^{-1}}{d\varepsilon}\bar{L}^{-1} + \bar{U}^{-1}\frac{d\bar{L}^{-1}}{d\varepsilon}\right)D_3 - D_2\bar{U}^{-1}\bar{L}^{-1}\frac{dD_3}{d\varepsilon}.$$

4.10 Numerical Results

We consider a full model and a reduced model in the network structure, obtaining optimal state and shape functionals with respect to the parameter ε .

The following material and geometric parameters apply to the beams considered. The elastic modulus is taken to be $\bar{E}_i = 2.1 \times 10^{11} \text{Nm}^{-2}$. The Poisson ratio ν_i is defined as

$$\nu_i = \frac{\bar{E}_i}{2\bar{G}_i} - 1, \text{ and we set } \nu_i = 0.3, \text{ which results in a shear modulus of } \bar{G}_i = 8.1 \times 10^{10} \text{Nm}^{-2}.$$

These values correspond to the material properties of steel, as stated in [15]. The cross-sectional area is $\bar{A}_i = 0.01 \text{m}^2$. The torsion constant and second moments of area is $\bar{I}_i^t = 1.41 \times 10^{-5} \text{m}^4$ and $\bar{I}_i^y = \bar{I}_i^z = 8.33 \times 10^{-6} \text{m}^4$, respectively, which are appropriate for a quadratic cross-section of the specified area. The shear coefficient that corresponds to

the chosen Poisson ratio and cross-sectional area is given by $\kappa = \frac{10(1 + \nu_i)}{12 + 11\nu_i} = 0.850$.

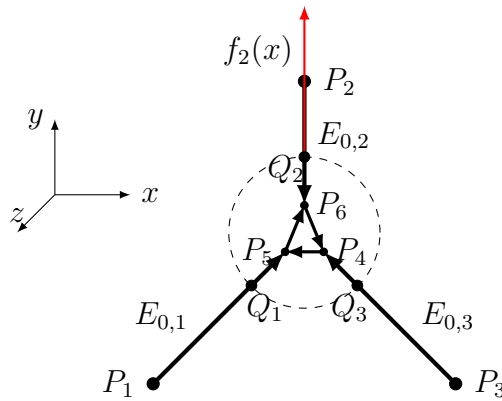
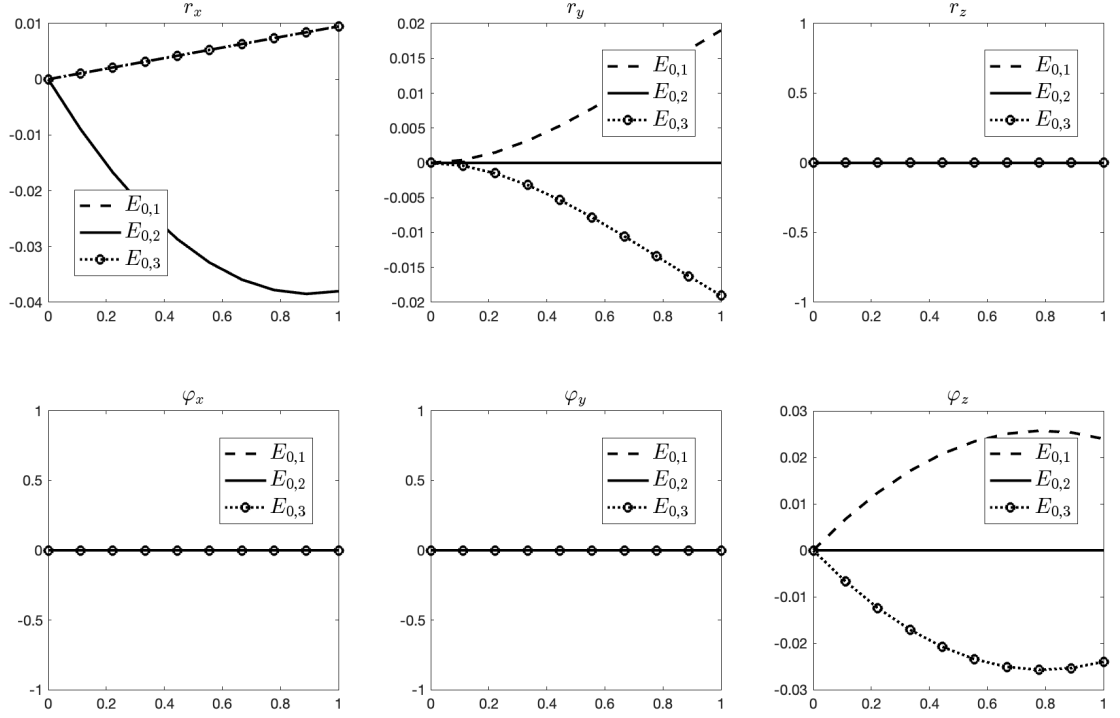


Figure 4.4: Timoshenko beams with cycle.

See Fig. 4.4 for the exact structure of the network. The vertex set consists of a small cycle with homogeneous Neumann boundary conditions, and three vertices (P_1 , P_2 ,

Figure 4.5: The static z of the full model in local coordinate.

and P_3) with homogeneous Dirichlet conditions. For this particular network, we have $\phi_i = (0, 0, \theta_i)^\top$, where the angle θ_i is defined as the angle from the x -coordinate to the edge $E_{\varepsilon,i}$ in the counter-clockwise direction. By (2.8), we have

$$\mathbb{E}_i = \mathbb{E}_i(\phi_i) = \begin{pmatrix} \cos \theta_i & -\sin \theta_i & 0 \\ \sin \theta_i & \cos \theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4.41)$$

Specifically, $\theta_1 = \frac{1}{6}\pi, \theta_2 = \frac{1}{2}\pi, \theta_3 = \frac{5}{6}\pi, \theta_4 = \pi, \theta_5 = \frac{1}{3}\pi, \theta_6 = -\frac{1}{3}\pi$. Set $L_i = 5$ ($i = 1, 2, 3$), $\varepsilon_0 = 0.5$ and $\varepsilon_{\max} = 1$. For v , there is a given distributed load in $E_{0,2}$, i.e., $v_2 = -10^8 \times (0, 2, 0, 0, 0, 0)^\top$. Then, we could get the exact expression of z by (4.20). Fig. 4.5 shows the z of the full model in local coordinates.

Now we will introduce the Hermite Finite Element Method. It is a high-order finite element method that can accurately solve the Timoshenko beam system. It employs Hermite polynomials as basis functions, which are polynomial functions with the property of having matching function values and derivatives at specified points.

On the unit interval $[0, 1]$, there exist four Hermite Shape Functions with cubic order, denoted by h_{00} , h_{01} , h_{10} , and h_{11} in Eq. (4.42). These functions are referred to as hat

functions on the reference unit, and they are graphically displayed in Fig. 4.6.

$$h_{00} = 2x^3 - 3x^2 + 1, h_{01} = -2x^3 + 3x^2, h_{10} = x^3 - 2x^2 + x, h_{11} = x^3 - x^2. \quad (4.42)$$

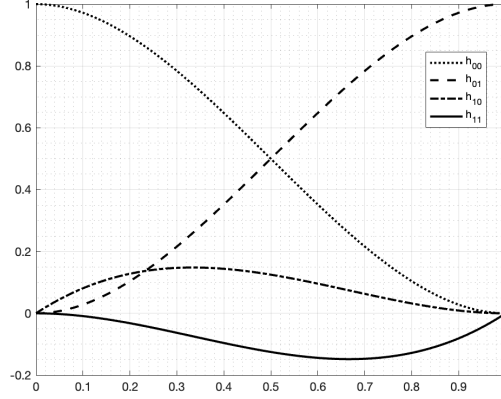


Figure 4.6: The four Hermite basis functions on the unit interval $[0, 1]$.

Let $0 = x_0 < x_1 < \dots < x_{N_i} = L_{0,i}$ be a local mesh on one edge interval $[0, L_{0,i}]$ which consisting of N_i subintervals of length h_i . After mapping from the reference unit to the local interval, the hat function can be given by $H_{1,i}$ and $H_{2,i}$ as follows:

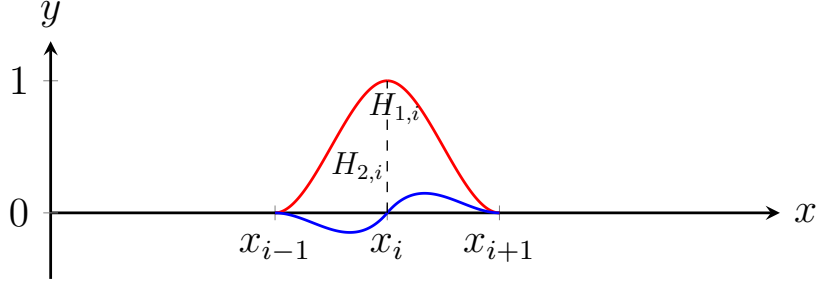
$$H_{1,i} = \begin{cases} -2 \left(\frac{x - x_{i-1}}{h_i} \right)^3 + 3 \left(\frac{x - x_{i-1}}{h_i} \right)^2 & \text{if } x \in [x_{i-1}, x_i]; \\ 2 \left(\frac{x - x_i}{h_i} \right)^3 - 3 \left(\frac{x - x_i}{h_i} \right)^2 + 1 & \text{if } x \in [x_i, x_{i+1}]; \\ 0 & \text{otherwise.} \end{cases}$$

$$H_{2,i} = \begin{cases} \left(\frac{x - x_{i-1}}{h_i} \right)^3 - \left(\frac{x - x_{i-1}}{h_i} \right)^2 & \text{if } x \in [x_{i-1}, x_i]; \\ \left(\frac{x - x_i}{h_i} \right)^3 - 2 \left(\frac{x - x_i}{h_i} \right)^2 + \frac{x - x_i}{h_i} & \text{if } x \in [x_i, x_{i+1}]; \\ 0 & \text{otherwise.} \end{cases}$$

Each hat function is continuous, piecewise cubic. For $H_{1,i}$, it takes a unit value at its own node x_i , while being zero at all other nodes. For $H_{2,i}$, its derivative takes a unit value at its own node x_i , while being zero at all other nodes (See Fig. 4.7).

The ansatz function $u_h(x)$ is constructed as a linear combination of two sets of basis functions, $H_{1,j}(x)$ and $H_{2,j}(x)$, that are defined over the set of nodes x_j .

$$u(x) \approx u_h(x) = \sum_{j=0}^{N_i} U_j H_{1,j}(x) + \sum_{j=0}^{N_i} \tilde{U}_j H_{2,j}(x).$$

Figure 4.7: Illustration of shape functions (red: $H_{1,i}$, blue: $H_{2,i}$)

At the j -th node x_j , there are two degrees of freedom U_j and $\tilde{U}_j$, where U_j is the value of $u_h(x)$ at x_j and $\tilde{U}_j$ is the corresponding derivative value. The discrete counterpart of the optimization control in Eq. (4.38) takes the form: find $u_h \in V_h$ and $p_h \in V_h$ such that

$$\begin{cases} (u_h, \varphi)_{\Omega_0} + a_0(p_h, \eta) - p_h(L-1)^\top \Lambda_\varepsilon \varphi(L-1) = (z, \varphi)_{\Omega_0}, \forall \varphi \in V_h \\ a_0(u_h, \varphi) - (p_h, \varphi)_{\Omega_0} - u_h(L-1)^\top \Lambda_\varepsilon \varphi(L-1) = (v, \varphi)_{\Omega_0}, \forall \varphi \in V_h \end{cases} \quad (4.43)$$

where u_h and p_h are the approximations of the solution and Lagrange multiplier, respectively. The function space V_h is defined as a set of cubic functions that satisfy certain conditions at the nodes and inner vertices.

$V_h = \{\varphi = (\varphi_1, \dots, \varphi_6) \mid \varphi_i \text{ is cubic on each } [x_j, x_{j+1}], i = 1, \dots, 6, \varphi_i(P_i) = 0, i = 1, 2, 3, \text{ continuity and Kirchhoff conditions at the inner vertices } P_4, P_5, P_6\}.$

All subscripts h represent discrete forms. Denote by $H_1 = (H_{1,1}, H_{1,2}, \dots, H_{1,N_i})^\top$, $H_2 = (H_{2,1}, H_{2,2}, \dots, H_{2,N_i})^\top$ be column vectors. The stiffness matrix K_i and mass matrix M_i could be written as

$$K_i = \begin{pmatrix} \int_0^{L_{0,i}} H'_1(H'_1)^\top & \int_0^{L_{0,i}} H'_2(H'_1)^\top \\ \int_0^{L_{0,i}} H'_1(H'_2)^\top & \int_0^{L_{0,i}} H'_2(H'_2)^\top \end{pmatrix}, \quad M_i = \begin{pmatrix} \int_0^{L_{0,i}} H_1(H_1)^\top & \int_0^{L_{0,i}} H_2(H_1)^\top \\ \int_0^{L_{0,i}} H_1(H_2)^\top & \int_0^{L_{0,i}} H_2(H_2)^\top \end{pmatrix},$$

We introduce another two matrix K_i^r and K_i^l :

$$K_i^r = \begin{pmatrix} \int_0^{L_{0,i}} H_1(H'_1)^\top & \int_0^{L_{0,i}} H_2(H'_1)^\top \\ \int_0^{L_{0,i}} H_1(H'_2)^\top & \int_0^{L_{0,i}} H_2(H'_2)^\top \end{pmatrix}, \quad K_i^l = \begin{pmatrix} \int_0^{L_{0,i}} H'_1(H_1)^\top & \int_0^{L_{0,i}} H'_2(H_1)^\top \\ \int_0^{L_{0,i}} H'_1(H_2)^\top & \int_0^{L_{0,i}} H'_2(H_2)^\top \end{pmatrix}.$$

Then we assemble the matrix

$$A_i = \begin{pmatrix} k^x K_i & 0 & 0 & 0 & 0 & 0 \\ 0 & k^s K_i & 0 & 0 & 0 & -k^s K_i^r \\ 0 & 0 & k^s K_i & 0 & k^s K_i^r & 0 \\ 0 & 0 & 0 & c^x K_i & 0 & 0 \\ 0 & 0 & k^s K_i^l & 0 & k^s M_i + c^y K_i & 0 \\ 0 & -k^s K_i^l & 0 & 0 & 0 & k^s M_i + c^z K_i \end{pmatrix},$$

and

$$B_i = \text{diag}(M_i, M_i, M_i, M_i, M_i, M_i)$$

is a block diagonal matrix. The matrix form of the Eq. (4.43) can be written as

$$\begin{pmatrix} A_1 & -B_1 & & & & \\ B_1 & A_1 & & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & A_6 & -B_6 \\ & & & & B_6 & A_6 \end{pmatrix} \begin{pmatrix} \bar{u}_1 \\ \bar{p}_1 \\ \vdots \\ \bar{u}_6 \\ \bar{p}_6 \end{pmatrix} = \begin{pmatrix} (\bar{v}_1, H) \\ (Z_1, H) \\ \vdots \\ (\bar{v}_6, H) \\ (Z_6, H) \end{pmatrix}, \quad (4.44)$$

where $\bar{u}_1 := \begin{pmatrix} (u_1, H_1) \\ (u'_1, H_1) \end{pmatrix}$, $(\bar{v}_1, H) := \begin{pmatrix} (v_1, H_1) \\ (v'_1, H_1) \end{pmatrix}$ and similar to others. In addition, we impose the coupling continuity and Kirchhoff condition.

Fig. 4.8 shows the state u in $E_{0,1}$, $E_{0,2}$ and $E_{0,3}$ in local coordinate for different ε comparing the desired state z .

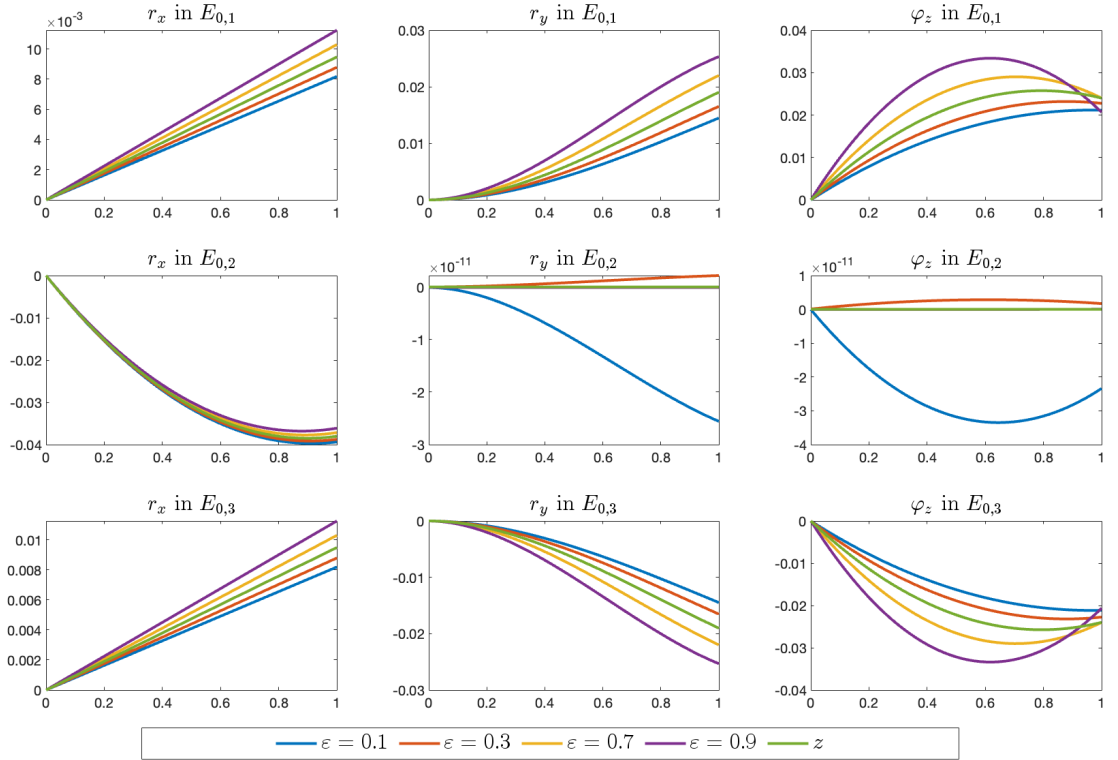


Figure 4.8: The state u for different $\varepsilon = 0.1, 0.3, 0.7, 0.9$ and z of full Timoshenko beam

Fig. 4.9 depicts the functional shape with respect to ε for the full and reduced models, respectively. It's easy to see that topological derivative is negative as $\varepsilon \rightarrow 0^+$. So the cost

functional is in fact reduced by the creation of a hole. The minimum value of J is when $\varepsilon = \varepsilon_0$.

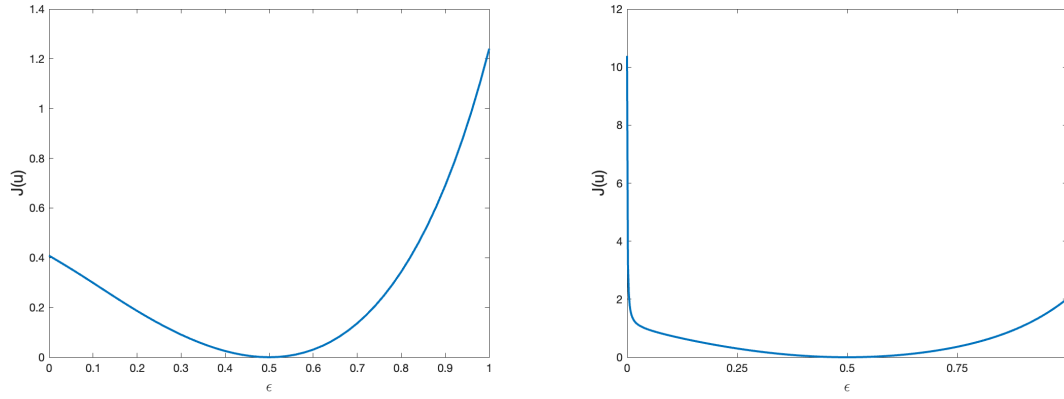


Figure 4.9: The shape functional for $\varepsilon \in [0, 1]$ of the full (left) and reduced (right) Timoshenko beam

We refer e.g., to [75], [74] for numerical methods in more complex geometry.

Chapter 5

Turnpike Property for Dynamic Optimization and Applications

The Turnpike Property for abstract wave equation is shown in [28]. The state equation is a scalar hyperbolic equation. The cost for control problems contains two components for the position y and the velocity y_t of the wave equation. The constant in the inequality (5.44) is uniform on compact sets of designs of the network. Therefore, the turnpike poverty holds for an admissible family of the networks. In such a way, the existence of optimum designs assured our control and shape optimization problems of the network by writing the static state equation.

In the proof, it is supposed that $y_t(T) = 0$, for the sake of simplicity. The turnpike property holds for the abstract control problem for the wave equation under the assumption that $y_t(T) = 0$.

5.1 Optimality System for Evolution Problem

We introduce the Lagrangian for the state $y \in L^2(0, T; H^1(\Omega))$, $y_t \in L^2(0, T; L^2(\Omega))$, $y_t(0) = y^1$ and the adjoint state $p \in L^2(0, T; H^1(\Omega))$, $p_t \in L^2(0, T; L^2(\Omega))$. Hence $y, p \in C(0, T; L^2(\Omega))$ and the integration by parts

$$\begin{aligned} & \int_0^T \left(\frac{\partial^2 y}{\partial^2 t}(t), p(t) \right)_{L^2(\Omega)} dt \\ &= - \int_0^T \left(\frac{\partial y}{\partial t}(t), \frac{\partial p}{\partial t}(t) \right)_{L^2(\Omega)} dt + \left(\frac{\partial y}{\partial t}(T), p(T) \right)_{L^2(\Omega)} - \left(\frac{\partial y}{\partial t}(0), p(0) \right)_{L^2(\Omega)} \\ &= - \int_0^T \left(\frac{\partial y}{\partial t}(t), \frac{\partial p}{\partial t}(t) \right)_{L^2(\Omega)} dt + \left(\frac{\partial y}{\partial t}(T), p(T) \right)_{L^2(\Omega)} - (y^1, p(0))_{L^2(\Omega)} \end{aligned}$$

Now, we assume $p(T) = 0$ which leads to the integration by parts formula

$$\int_0^T \left(\frac{\partial^2 y}{\partial^2 t}(t), p(t) \right)_{L^2(\Omega)} dt = - \int_0^T \left(\frac{\partial y}{\partial t}(t), \frac{\partial p}{\partial t}(t) \right)_{L^2(\Omega)} dt - (y^1, p(0))_{L^2(\Omega)}$$

Remark 5. *If we add the state constraint $y_t(T) = 0$ then the integration by parts formula is the same but the terminal condition for $p(T)$ becomes undetermined, i.e., we lose homogeneous condition for the terminal adjoint state.*

We derive the optimality system for the optimal control problem with the evolution state equation. To this end, we introduce the Lagrangian with $y \in L^2(0, T; H^1(\Omega))$, $y_t \in L^2(0, T; \Omega)$, $y(0) = y^0$, and $\varphi \in L^2(0, T; H^1(\Omega))$, $\varphi_t \in L^2(0, T; \Omega)$, $\varphi(T) = 0$

$$\begin{aligned} \mathcal{L}(u, y, \varphi) = & \frac{1}{2} \int_0^T \|y - y^d\|_{L^2(\Omega)}^2 dt + \frac{1}{2} \int_0^T \|u - u^d\|_{L^2(\Gamma)}^2 dt \\ & + \frac{\gamma}{2} \int_0^T \|\partial_t(y - y^d)\|_{L^2(\Omega)}^2 dt \\ & - \int_0^T \left(\frac{\partial y}{\partial t}(t), \frac{\partial \varphi}{\partial t}(t) \right)_{L^2(\Omega)} dt - (y^1, \varphi(0))_{L^2(\Omega)} + \int_0^T a(y(t), \varphi) dt \\ & - \int_0^T (L(u)(t), \varphi)_{L^2(\Gamma)} dt - \int_0^T (F(t), \varphi)_{L^2(\Omega)} dt. \end{aligned}$$

The adjoint state $p = p(u; x, t)$ is obtained by differentiation of Lagrangian with respect to the state, thus with $Q(T) = (0, T) \times \Omega$

$$\begin{aligned} (p_{tt}, \varphi)_{Q(T)} + \int_0^T a(p, \varphi) dt &= (y^d - y, \varphi)_{Q(T)} + \gamma (\partial_t(y^d - y), \varphi_t)_{Q(T)} \quad \forall \varphi \in H(Q(T)) \\ p(T) = p_t(T) &= 0. \end{aligned}$$

The gradient of the cost with respect to the control is obtained by differentiation of the Lagrangian with respect to the control

$$\begin{aligned} \left\langle \frac{dJ}{du}(u), \eta \right\rangle &= \int_0^T (u - u^d, \eta)_{L^2(\Gamma)} dt - \int_0^T (L(\eta)(t), p(t))_{L^2(\Gamma)} dt \\ &= \int_0^T (u - u^d, \eta)_{\Gamma} dt - \int_0^T (L(\eta)(t), p(t))_{\Gamma} dt. \end{aligned}$$

The following lemma contains the necessary optimality conditions for the dynamic problem where $J_T(u)$ as defined in (4.3) is minimized.

Lemma 1. *The optimality system for the optimal control of the evolution control problem is verified for a.e. $t \in (0, T)$:*

$$(\hat{y}_{tt}, \varphi)_{Q(T)} + \int_0^T a(\hat{y}(t), \varphi) dt = \int_0^T (L(\hat{u})(t), \varphi)_{\Gamma} dt + (F(t), \varphi)_{Q(T)} \quad (5.1)$$

$$\begin{aligned} & \forall \varphi \in H(Q(T)) \\ \hat{y}(0) = y^0, \quad \hat{y}_t(0) &= y^1 \end{aligned} \quad (5.2)$$

$$(\hat{p}_{tt}, \varphi)_{H(Q(T))} + \int_0^T a(\hat{p}, \varphi) dt = (y^d - \hat{y} + \gamma \hat{y}_{tt}, \varphi)_{Q(T)} \quad (5.3)$$

$$\forall \varphi \in H(Q(T))$$

$$\hat{p}(T) = \hat{p}_t(T) = 0 \quad (5.4)$$

$$(\hat{u} - u^d, \eta)_\Gamma - (L(\eta)(t), \hat{p}(t))_\Gamma = 0 \quad \forall \eta \in L^2(0, T; \Gamma) \quad (5.5)$$

The optimality system (5.1)-(5.5) admits a unique solution $(\hat{u}, \hat{y}, \hat{p})$.

5.2 The difference of the Static and the Dynamic Optimality systems for Distributed Control

In this section, we study a system that is satisfied by the ordered pair that has the difference between the optimal dynamic state for the time horizon T and the optimal static state as the first component and the difference between the optimal dynamic adjoint state for the time horizon T and the optimal static adjoint state as the second component. The question of long time versus steady state optimal control has already been studied in [67] where the focus is on the turnpike property of the state and the control without the adjoint state. The turnpike phenomenon for optimal boundary control problems with first order hyperbolic systems is considered in [23].

We assume that $\Gamma = \Omega$ and the operators $L(v)$ and $L(u)$ are identity operators in $L^2(\Omega)$ and $F(t) = f$, $u^d(t) = v_d$, $y^d = z_d$ for $t \in [0, T]$ almost everywhere. The optimality system for the static problem reads

$$\hat{z}^\sigma \in H : a(\hat{z}^\sigma, \varphi) = (\hat{v}^\sigma, \varphi)_{L^2(\Omega)} + (f, \varphi)_{L^2(\Omega)} \quad \forall \varphi \in H, \quad (5.6)$$

$$\hat{p}^\sigma \in H : a(\hat{p}^\sigma, \phi) = (z_d - \hat{z}^\sigma, \phi)_{L^2(\Omega)} \quad \forall \phi \in H, \quad (5.7)$$

$$\hat{v}^\sigma - v_d = \hat{p}^\sigma \text{ a.e. in } \Omega. \quad (5.8)$$

The optimality system for the evolution problem implies for $t \in [0, T]$ almost everywhere (with $F(t) = f$)

$$(\hat{y}_{tt}^T(t), \varphi)_{L^2(\Omega)} + a(\hat{y}^T(t), \varphi) = (\hat{u}^T(t), \varphi)_{L^2(\Omega)} + (F(t), \varphi)_{L^2(\Omega)} \quad \forall \varphi \in H \quad (5.9)$$

$$\hat{y}^T(0) = y^0, \quad \hat{y}_t^T(0) = y^1 \quad (5.10)$$

$$(\hat{p}_{tt}^T(t), \varphi)_{L^2(\Omega)} + a(\hat{p}^T(t), \varphi) = (y^d - \hat{y}^T(t) + \gamma \hat{y}_{tt}^T(t), \varphi)_{L^2(\Omega)} \quad \forall \varphi \in H \quad (5.11)$$

$$\hat{p}^T(T) = \hat{p}_t^T(T) = 0 \quad (5.12)$$

$$\hat{u}^T(t) - u^d = \hat{p}^T(t) \text{ a.e. in } \Omega \times (0, T). \quad (5.13)$$

Define the differences

$$\omega^T = \hat{y}^T - \hat{z}^\sigma, \quad \mu^T = \hat{p}^T - \hat{p}^\sigma, \quad \nu^T = \hat{u}^T - \hat{v}^\sigma.$$

Then for all $\varphi \in H$ we have the initial condition

$$\omega^T(0) = y^0 - \hat{z}^\sigma, \quad \omega_t^T(0) = y^1, \quad (5.14)$$

the terminal conditions

$$\mu^T(T) = -\hat{p}^\sigma, \quad \mu_t^T(T) = 0, \quad (5.15)$$

the dynamics

$$(\omega_{tt}^T(t), \varphi)_{L^2(\Omega)} + a(\omega^T(t), \varphi) = (\nu^T(t), \varphi)_{L^2(\Omega)} = (\mu^T(t), \varphi)_{L^2(\Omega)} \quad (5.16)$$

and with the assumption that $y^d = z_d$ and $u^d = v_d$

$$(\mu_{tt}^T(t), \varphi)_{L^2(\Omega)} + a(\mu^T(t), \varphi) = -(\omega^T(t), \varphi)_{L^2(\Omega)} + \gamma (\omega_{tt}^T(t), \varphi)_{L^2(\Omega)}. \quad (5.17)$$

Note that for the difference system, the existence of a solution is implied by the construction as the difference between two systems, for which solutions exist.

Note that for the energy

$$E(t) = \frac{(\omega_t^T(t), \omega_t^T(t))_{L^2(\Omega)} + a(\omega^T(t), \omega^T(t))}{2} + \frac{(\mu_t^T(t), \mu_t^T(t))_{L^2(\Omega)} + a(\mu^T(t), \mu^T(t))}{2}$$

due to (5.16) and (5.17) for $\gamma = 0$ we have

$$E''(t) \geq (\omega^T(t), \omega^T(t))_{L^2(\Omega)} + (\mu^T(t), \mu^T(t))_{L^2(\Omega)} \geq 0.$$

Thus E is convex on $[0, T]$.

Now we perform a spectral analysis to show the exponential turnpike property.

Assume that there exists a complete orthonormal sequence $(\psi_k)_{k=1}^\infty$ of eigenfunctions with $a(\psi_k, \varphi) = \lambda_k(\psi_k, \varphi)_{L^2(\Omega)}$ for all $k \in \{0, 1, 2, 3, \dots\}$ where

$$\lambda_k \geq \gamma > 0 \quad (5.18)$$

is a real number.

Remark 6. *In the case of optimal design, the bilinear form depends on Ω . In the case of a graph, this means the dependence of the lengths of the edges. In this case, the eigenvalues and eigenfunctions depend on these parameters that we denote by ℓ . Therefore a meaningful analysis has to take into account the sensitivity with respect to ℓ .*

Our assumption on the feasible designs is that the smallest eigenvalues are greater than or equal to the given strictly positive lower-bound $\gamma > 0$ uniformly on the set of admissible designs. In this way, we ensure that the turnpike property is valid for the bilevel optimization problem that we consider in this paper.

It is well known that the smallest eigenvalue that can be characterized as the Rayleigh quotient depends smoothly on the parameters, see [29]. In our analysis, the particular structure of the spectrum is not relevant.

Then we can use the representations

$$\omega^T = \sum_{k=0}^{\infty} a_k(t) \psi_k(x), \quad \mu^T = \sum_{k=0}^{\infty} b_k(t) \psi_k(x) \quad (5.19)$$

to show that ω^T and μ^T have the turnpike property.

It is clear that the functions a_k and b_k depend on T as a parameter, so a more precise notation would be $a_{k,T}(t)$ and $b_{k,T}(t)$. However, in order to make the text more concise we continue with the shorter notation a_k and b_k .

From (5.16) and (5.17) we obtain $a_k'' = -\lambda_k a_k + b_k$ and $b_k'' = -\lambda_k b_k - a_k + \gamma a_k''$. Thus we have the sequence of differential equations

$$\begin{aligned} a_k^{(4)} + (2\lambda_k - \gamma) a_k^{(2)} + (\lambda_k^2 + 1) a_k &= 0, \\ b_k^{(4)} + (2\lambda_k - \gamma) b_k^{(2)} + (\lambda_k^2 + 1) b_k &= 0. \end{aligned} \quad (5.20)$$

Define the characteristic polynomial

$$p_k(z) = z^4 + (2\lambda_k - \gamma) z^2 + (\lambda_k^2 + 1).$$

Since p_k is a polynomial in z^2 , for the roots $z_k^{(l)}$ of p_k we obtain

$$(z_k^{(l)})^2 = \frac{\gamma}{2} - \lambda_k \pm \frac{i}{2} \sqrt{4 + 4\lambda_k \gamma - \gamma^2}$$

for $l \in \{1, 2, 3, 4\}$ which implies $|(z_k^{(l)})^2| = \sqrt{1 + \lambda_k^2}$. Thus there are two pairs of complex conjugate roots and we have the representation

$$p_k(z) = (z - z_k^{(1)})(z - \overline{z_k^{(1)}})(z + z_k^{(1)})(z + \overline{z_k^{(1)}}).$$

We have $|z_k^{(l)}|^4 = \lambda_k^2 + 1$, $\operatorname{Re}((z_k^{(l)})^2) = \frac{\gamma}{2} - \lambda_k$ and $|\operatorname{Im}((z_k^{(l)})^2)| = \frac{\sqrt{4 + 4\lambda_k \gamma - \gamma^2}}{2}$. Moreover, we have

$$|\operatorname{Re}(z_k^{(l)})| = \sqrt{\frac{\gamma}{4} + \frac{1}{2(\lambda_k + \sqrt{1 + \lambda_k^2})}}$$

and

$$|\operatorname{Re}(z_k^{(l)})| \geq \frac{\sqrt{\gamma}}{2}. \quad (5.21)$$

The initial condition (5.14) yields the values for $a_k(0)$ and $a_k'(0)$. The terminal condition (5.15) yields the value for $b_k(T)$ and $b_k'(T) = 0$. Note that (5.15) implies that the value of $b_k(T)$ is also independent of T , just like $b_k'(T)$.

For the sake of conciseness, in the sequel we use the notation $z_k = z_k^{(1)}$.

5.3 Representation of the Solution

The solution of the optimality system means that for the coefficients $b_{k,T}$ of μ^T as defined in (5.19) we solve a boundary value problem with the ODE of order four (5.20) i.e.

$$\left\{ \begin{array}{ll} b_{k,T}^{(4)} + (2\lambda_k - \gamma) b_{k,T}^{(2)} + (\lambda_k^2 + 1) b_{k,T} &= 0 \\ b_{k,T}(T) &= \beta_k \\ b_{k,T}'(T) &= 0 \\ -b_{k,T}''(0) + (\gamma - \lambda_k) b_{k,T}(0) &= (1 + \gamma \lambda_k) a_{k,T}(0) \\ -b_{k,T}'''(0) + (\gamma - \lambda_k) b_{k,T}'(0) &= (1 + \gamma \lambda_k) a_{k,T}'(0), \end{array} \right. \quad (5.22)$$

where the value of β_k is determined by the terminal condition $\mu^T(T) = -\hat{p}^\sigma$ in (5.15). We represent the solution in the form

$$b_{k,T}(t) = F_{k,T}(t) \beta_k + (1 + \gamma \lambda_k)[G_{k,T}(t) a_{k,T}(0) + H_{k,T}(t) a'_{k,T}(0)]. \quad (5.23)$$

The following Lemma contains explicit representations of $F_{k,T}$, $G_{k,T}$ and $H_{k,T}$. In the representation, the numbers $d(k, T)$ appear as multipliers, therefore it is important that for T sufficiently large we have $d(k, T) \neq 0$. Since in the study of the turnpike phenomenon we are interested in large time horizons, the assumption that the time horizon T is large is not restrictive for us. We introduce the notation

$$\Xi_k := \gamma - \lambda_k - z_k^2 = \frac{\gamma}{2} \mp \frac{i}{2} \sqrt{4 + 4 \lambda_k \gamma - \gamma^2}. \quad (5.24)$$

Lemma 2. For $k \in \{0, 1, 2, \dots\}$ and T sufficiently large define the numbers

$$d(k, T) = -2 \operatorname{Re}(\Xi_k^2) + 2|\Xi_k|^2 \operatorname{Re} \left(\cosh(\overline{z_k} T) \cosh(z_k T) - \frac{\overline{z_k}}{z_k} \sinh(\overline{z_k} T) \sinh(z_k T) \right). \quad (5.25)$$

Then we have

$$\begin{aligned} & d(k, T) F_{k,T}(t) \\ &= 2 \operatorname{Re} \left(\left[-\Xi_k^2 + |\Xi_k|^2 \cosh(z_k T) \cosh(\overline{z_k} T) \right] \cosh(z_k (t - T)) \right) \\ &+ 2 \operatorname{Re} (|\Xi_k|^2 \cosh(\overline{z_k} T) \sinh(z_k T) \sinh(z_k (t - T))) \\ &- 2|\Xi_k|^2 \operatorname{Re} \left(\frac{\overline{z_k}}{z_k} \sinh(\overline{z_k} T) [\sinh(z_k T) \cosh(z_k (t - T)) + \cosh(z_k T) \sinh(z_k (t - T))] \right). \end{aligned}$$

Furthermore, it holds

$$\begin{aligned} & d(k, T) G_{k,T}(t) \\ &= -2 \operatorname{Re} (\Xi_k [\cosh(z_k T) \cosh(z_k (t - T)) + \sinh(z_k T) \sinh(z_k (t - T))]) \\ &+ 2 \operatorname{Re} \left(\overline{\Xi_k} \left[\cosh(\overline{z_k} T) \cosh(z_k (t - T)) + \frac{\overline{z_k}}{z_k} \sinh(\overline{z_k} T) \sinh(z_k (t - T)) \right] \right) \end{aligned} \quad (5.26)$$

and

$$\begin{aligned} & d(k, T) H_{k,T}(t) \\ &= 2 \operatorname{Re} \left(\frac{\overline{\Xi_k}}{\overline{z_k}} \sinh(\overline{z_k} T) \cosh(z_k (t - T)) + \frac{\overline{\Xi_k}}{z_k} \cosh(\overline{z_k} T) \sinh(z_k (t - T)) \right) \\ &- 2 \operatorname{Re} \left(\frac{\Xi_k}{z_k} [\sinh(z_k T) \cosh(z_k (t - T)) + \cosh(z_k T) \sinh(z_k (t - T))] \right). \end{aligned}$$

Proof: Due to the properties of the hyperbolic functions that can be found for example in [8, 9] we have

$$\begin{aligned} d(k, T) &= -2 \operatorname{Re}(\Xi_k^2) \\ &+ |\Xi_k|^2 \left[\left(1 + \frac{1 - \frac{\gamma}{2\lambda_k}}{\sqrt{1 + \frac{1}{\lambda_k^2}}} \right) \cosh(2 \operatorname{Re}(z_k) T) + \left(1 - \frac{1 - \frac{\gamma}{2\lambda_k}}{\sqrt{1 + \frac{1}{\lambda_k^2}}} \right) \cos(2 \operatorname{Im}(z_k) T) \right] \end{aligned} \quad (5.27)$$

This implies $\lim_{T \rightarrow \infty} |d(k, T)| = \infty$, hence for all sufficiently large T we have $d(k, T) \neq 0$. In the remaining part of the proof, we assume that $d(k, T) \neq 0$. Moreover, we see that $d(k, T)$ grows exponentially fast with T with the growth rate

$$2\operatorname{Re}(z_k) \geq \sqrt{\gamma} \quad (5.28)$$

due to (5.21).

To be more precise, note that $\operatorname{Re}(\Xi_k^2) = \frac{\gamma^2}{2} - 1 + \frac{5}{2}\gamma\lambda_k + \lambda_k^2$ grows with k with the order λ_k^2 and $|\Xi_k|^2 = 1 + \gamma\lambda_k$ grows with the order λ_k .

At this point, elementary computations show that $F_{k,T}(t)$, $G_{k,T}(t)$ and $H_{k,T}(t)$ satisfy the differential equation and the boundary conditions where $(\beta_k, a_{k,T}(0), a'_{k,T}(0))$ have the values $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ respectively. Now we present details of the verification of these basic functions.

Verification of the Basis Function $F_{k,T}(t)$ We start with the function $F_{k,T}(t)$ from (5.23) that is multiplied with β_k , i.e. $d(k, T) F_{k,T}(t)$. We have

$$\begin{aligned} d(k, T) F_{k,T}(t) &= 2 \operatorname{Re} \left(\left[-\overline{\Xi_k}^2 + |\Xi_k|^2 \cosh(z_k T) \cosh(\overline{z_k} T) \right] \cosh(z_k (t - T)) \right) \\ &\quad + 2 \operatorname{Re} (|\Xi_k|^2 \cosh(\overline{z_k} T) \sinh(z_k T) \sinh(z_k (t - T))) \\ &\quad - 2 \operatorname{Re} \left(\frac{\overline{z_k}}{z_k} |\Xi_k|^2 \sinh(z_k T) \sinh(\overline{z_k} T) \cosh(z_k (t - T)) \right) \\ &\quad - 2 \operatorname{Re} \left(\frac{\overline{z_k}}{z_k} |\Xi_k|^2 \cosh(z_k T) \sinh(\overline{z_k} T) \sinh(z_k (t - T)) \right). \end{aligned}$$

Hence for $t = T$ we have

$$\begin{aligned} d(k, T) F_{k,T}(T) &= 2 \operatorname{Re} \left(-\overline{\Xi_k}^2 + |\Xi_k|^2 \cosh(z_k T) \cosh(\overline{z_k} T) \right) \\ &= -2 \operatorname{Re} (\Xi_k^2) + 2 |\Xi_k|^2 \operatorname{Re} \left(\cosh(\overline{z_k} T) \cosh(z_k T) - \frac{\overline{z_k}}{z_k} \sinh(\overline{z_k} T) \sinh(z_k T) \right) \\ &= d(k, T). \end{aligned}$$

Since $d(k, T) \neq 0$, this implies $F_{k,T}(T) = 1$. For the derivative we obtain

$$\begin{aligned} d(k, T) F'_{k,T}(t) &= 2 \operatorname{Re} \left(z_k \left[-\overline{\Xi_k}^2 + |\Xi_k|^2 [\cosh(z_k T) \cosh(\overline{z_k} T)] \right] \sinh(z_k (t - T)) \right) \\ &\quad + 2 \operatorname{Re} (z_k |\Xi_k|^2 \cosh(\overline{z_k} T) \sinh(z_k T) \cosh(z_k (t - T))) \\ &\quad - 2 \operatorname{Re} (\overline{z_k} |\Xi_k|^2 \sinh(z_k T) \sinh(\overline{z_k} T) \sinh(z_k (t - T))) \\ &\quad - 2 \operatorname{Re} (\overline{z_k} |\Xi_k|^2 \cosh(z_k T) \sinh(\overline{z_k} T) \cosh(z_k (t - T))). \end{aligned}$$

Thus we have

$$d(k, T) F'_{k,T}(T) = 2 \operatorname{Re} (z_k |\Xi_k|^2 \cosh(\overline{z_k} T) \sinh(z_k T)) - 2 \operatorname{Re} (\overline{z_k} |\Xi_k|^2 \cosh(z_k T) \sinh(\overline{z_k} T)) \\ = 0.$$

Therefore we have $F'_{k,T}(T) = 0$. For $t = 0$ we have

$$d(k, T) F_{k,T}(0) = 2 \operatorname{Re} \left(\left[-\overline{\Xi_k}^2 + |\Xi_k|^2 [\cosh(z_k T) \cosh(\overline{z_k} T)] \right] \cosh(z_k T) \right) \\ - 2 \operatorname{Re} (|\Xi_k|^2 \cosh(\overline{z_k} T) \sinh(z_k T) \sinh(z_k T)) \\ = 2 \operatorname{Re} \left(-\overline{\Xi_k}^2 \cosh(z_k T) + |\Xi_k|^2 \cosh(\overline{z_k} T) [\cosh(z_k T)^2 - \sinh^2(z_k T)] \right) \\ = 2 \operatorname{Re} \left(-\overline{\Xi_k}^2 \cosh(z_k T) + |\Xi_k|^2 \cosh(\overline{z_k} T) \right).$$

For the second derivative, we have

$$d(k, T) F''_{k,T}(t) = 2 \operatorname{Re} \left(z_k^2 \left[-\overline{\Xi_k}^2 + |\Xi_k|^2 [\cosh(z_k T) \cosh(\overline{z_k} T)] \right] \cosh(z_k (t - T)) \right) \\ + 2 \operatorname{Re} (z_k^2 |\Xi_k|^2 \cosh(\overline{z_k} T) \sinh(z_k T) \sinh(z_k (t - T))) \\ - 2 \operatorname{Re} (z_k \overline{z_k} |\Xi_k|^2 \sinh(z_k T) \sinh(\overline{z_k} T) \cosh(z_k (t - T))) \\ - 2 \operatorname{Re} (z_k \overline{z_k} |\Xi_k|^2 \cosh(z_k T) \sinh(\overline{z_k} T) \sinh(z_k (t - T))).$$

For $t = 0$ this yields

$$d(k, T) F''_{k,T}(0) = 2 \operatorname{Re} \left(z_k^2 \left[-\overline{\Xi_k}^2 + |\Xi_k|^2 \cosh(z_k T) \cosh(\overline{z_k} T) \right] \cosh(z_k T) \right) \\ - 2 \operatorname{Re} (z_k^2 |\Xi_k|^2 \cosh(\overline{z_k} T) \sinh(z_k T) \sinh(z_k T)) \\ = 2 \operatorname{Re} \left(z_k^2 \left[-\overline{\Xi_k}^2 \cosh(z_k T) + |\Xi_k|^2 \cosh(\overline{z_k} T) \right] \right).$$

Hence we have

$$d(k, T) [-F''_{k,T}(0) + (\gamma - \lambda_k) F_{k,T}(0)] \\ = 2 \operatorname{Re} \left(\Xi_k \left[-\overline{\Xi_k}^2 \cosh(z_k T) + |\Xi_k|^2 \cosh(\overline{z_k} T) \right] \right) \\ = 2 |\Xi_k|^2 \operatorname{Re} ([-\overline{\Xi_k} \cosh(z_k T) + \Xi_k \cosh(\overline{z_k} T)]) \\ = 0.$$

Thus we have $-F''_{k,T}(0) + (\gamma - \lambda_k) F_{k,T}(0) = 0$.

To proceed, let us observe that

$$d(k, T) F'_{k,T}(0) = 2 \operatorname{Re} \left(z_k \left[\overline{\Xi_k}^2 - |\Xi_k|^2 \cosh(z_k T) \cosh(\overline{z_k} T) \right] \sinh(z_k T) \right) \\ + 2 \operatorname{Re} (z_k |\Xi_k|^2 \cosh(\overline{z_k} T) \sinh(z_k T) \cosh(z_k T)) \\ + 2 \operatorname{Re} (\overline{z_k} |\Xi_k|^2 \sinh(z_k T) \sinh(\overline{z_k} T) \sinh(z_k T))$$

$$\begin{aligned}
& -2 \operatorname{Re} \left(\overline{z_k} |\Xi_k|^2 \cosh(z_k T) \sinh(\overline{z_k} T) \cosh(z_k T) \right) \\
& = 2 \operatorname{Re} \left(z_k \overline{\Xi_k}^2 \sinh(z_k T) \right) - 2 \operatorname{Re} \left(\overline{z_k} |\Xi_k|^2 \sinh(\overline{z_k} T) \right).
\end{aligned}$$

For the third derivative, we have

$$\begin{aligned}
d(k, T) F_{k,T}'''(t) &= 2 \operatorname{Re} \left(z_k^3 \left[-\overline{\Xi_k}^2 + |\Xi_k|^2 \cosh(z_k T) \cosh(\overline{z_k} T) \right] \sinh(z_k(t - T)) \right) \\
&+ 2 \operatorname{Re} \left(z_k^3 |\Xi_k|^2 \cosh(\overline{z_k} T) \sinh(z_k T) \cosh(z_k(t - T)) \right) \\
&- 2 \operatorname{Re} \left(z_k^2 \overline{z_k} |\Xi_k|^2 \sinh(z_k T) \sinh(\overline{z_k} T) \sinh(z_k(t - T)) \right) \\
&- 2 \operatorname{Re} \left(z_k^2 \overline{z_k} |\Xi_k|^2 \cosh(z_k T) \sinh(\overline{z_k} T) \cosh(z_k(t - T)) \right).
\end{aligned}$$

This yields

$$\begin{aligned}
d(k, T) F_{k,T}'''(0) &= -2 \operatorname{Re} \left(z_k^3 \left[-\overline{\Xi_k}^2 + |\Xi_k|^2 \cosh(z_k T) \cosh(\overline{z_k} T) \right] \sinh(z_k T) \right) \\
&+ 2 \operatorname{Re} \left(z_k^3 |\Xi_k|^2 \cosh(\overline{z_k} T) \sinh(z_k T) \cosh(z_k T) \right) \\
&- 2 \operatorname{Re} \left(z_k^2 \overline{z_k} |\Xi_k|^2 \sinh(\overline{z_k} T) \right) \\
&= 2 \operatorname{Re} \left(z_k^3 \overline{\Xi_k}^2 \sinh(z_k T) \right) - 2 \operatorname{Re} \left(z_k^2 \overline{z_k} |\Xi_k|^2 \sinh(\overline{z_k} T) \right).
\end{aligned}$$

Hence we have

$$\begin{aligned}
d(k, T) \left[-F_{k,T}'''(0) + (\gamma - \lambda_k) F_{k,T}'(0) \right] &= -2 \operatorname{Re} \left(z_k^3 \overline{\Xi_k}^2 \sinh(z_k T) \right) + 2 \operatorname{Re} \left(z_k^2 \overline{z_k} |\Xi_k|^2 \sinh(\overline{z_k} T) \right) \\
&+ (\gamma - \lambda_k) 2 \operatorname{Re} \left(z_k \overline{\Xi_k}^2 \sinh(z_k T) \right) - (\gamma - \lambda_k) 2 \operatorname{Re} \left(\overline{z_k} |\Xi_k|^2 \sinh(\overline{z_k} T) \right) \\
&= 2 \operatorname{Re} \left(z_k \overline{\Xi_k} \overline{\Xi_k}^2 \sinh(z_k T) \right) - 2 \operatorname{Re} \left(\overline{\Xi_k} \overline{z_k} |\Xi_k|^2 \sinh(\overline{z_k} T) \right) \\
&= 0.
\end{aligned}$$

Thus we have shown that

$$\begin{aligned}
F_{k,T}(T) &= 1, \\
F_{k,T}'(T) &= 0, \\
-F_{k,T}''(0) + (\gamma - \lambda_k) F_{k,T}'(0) &= 0, \\
-F_{k,T}'''(0) + (\gamma - \lambda_k) F_{k,T}'(0) &= 0.
\end{aligned}$$

Verification of the Basis Function $G_{k,T}(t)$. We continue with $G_{k,T}(t)$ from (5.23) that is multiplied with $(1 + \gamma \lambda_k) a_{k,T}(0)$, i.e.

$$\begin{aligned}
d(k, T) G_{k,T}(t) &= -2 \operatorname{Re} \left(\overline{\Xi_k} \cosh(z_k T) [1 + \tanh(z_k T) \tanh(z_k(t - T))] \cosh(z_k(t - T)) \right) \\
&+ 2 \operatorname{Re} \left(\overline{\Xi_k} \cosh(\overline{z_k} T) \left[1 + \frac{\overline{z_k}}{z_k} \tanh(\overline{z_k} T) \tanh(z_k(t - T)) \right] \cosh(z_k(t - T)) \right).
\end{aligned}$$

Due to the definition of z_k , the function $G_{k,T}(t)$ satisfies the ODE (5.20).

Moreover, for $t = T$ we have

$$d(k, T) G_{k,T}(T) = -2 \operatorname{Re} (\Xi_k \cosh(z_k T)) + 2 \operatorname{Re} (\overline{\Xi_k} \cosh(\overline{z_k} T)) = 0.$$

In addition, for the derivative we obtain

$$\begin{aligned} d(k, T) G'_{k,T}(t) &= -2 \operatorname{Re} (\Xi_k z_k [\cosh(z_k T) \sinh(z_k(t - T)) + \sinh(z_k T) \cosh(z_k(t - T))]) \\ &\quad + 2 \operatorname{Re} (\overline{\Xi_k} [z_k \cosh(\overline{z_k} T) \sinh(z_k(t - T)) + \overline{z_k} \sinh(\overline{z_k} T) \cosh(z_k(t - T))]). \end{aligned}$$

Hence for $t = T$, we have the derivative

$$d(k, T) G'_{k,T}(T) = -2 \operatorname{Re} (\Xi_k z_k \sinh(z_k T)) + 2 \operatorname{Re} (\overline{\Xi_k} \overline{z_k} \sinh(\overline{z_k} T)) = 0.$$

For the second derivative, we obtain

$$\begin{aligned} d(k, T) G''_{k,T}(t) &= -2 \operatorname{Re} (\Xi_k z_k^2 [\cosh(z_k T) \cosh(z_k(t - T)) + \sinh(z_k T) \sinh(z_k(t - T))]) \\ &\quad + 2 \operatorname{Re} (\overline{\Xi_k} z_k [z_k \cosh(\overline{z_k} T) \cosh(z_k(t - T)) + \overline{z_k} \sinh(\overline{z_k} T) \sinh(z_k(t - T))]). \end{aligned}$$

For $t = 0$ this implies

$$\begin{aligned} d(k, T) G''_{k,T}(0) &= -2 \operatorname{Re} (\Xi_k z_k^2 [\cosh^2(z_k T) - \sinh^2(z_k T)]) \\ &\quad + 2 \operatorname{Re} (\overline{\Xi_k} z_k [z_k \cosh(\overline{z_k} T) \cosh(z_k T) - \overline{z_k} \sinh(\overline{z_k} T) \sinh(z_k T)]) \\ &= -2 \operatorname{Re} (\Xi_k z_k^2) + 2 \operatorname{Re} (\overline{\Xi_k} z_k [z_k \cosh(\overline{z_k} T) \cosh(z_k T) - \overline{z_k} \sinh(\overline{z_k} T) \sinh(z_k T)]). \end{aligned}$$

For $t = 0$, equation (5.26) implies

$$d(k, T) G_{k,T}(0) = -2 \operatorname{Re} (\Xi_k) + 2 \operatorname{Re} \left(\overline{\Xi_k} \left[\cosh(\overline{z_k} T) \cosh(z_k T) - \frac{\overline{z_k}}{z_k} \sinh(\overline{z_k} T) \sinh(z_k T) \right] \right).$$

Hence in view of (5.24) we obtain

$$\begin{aligned} d(k, T) [-G''_{k,T}(0) + (\gamma - \lambda_k) G_{k,T}(0)] &= 2 \operatorname{Re} (\Xi_k z_k^2) - 2 \operatorname{Re} (\overline{\Xi_k} z_k [z_k \cosh(\overline{z_k} T) \cosh(z_k T) - \overline{z_k} \sinh(\overline{z_k} T) \sinh(z_k T)]) \\ &\quad - (\gamma - \lambda_k) 2 \operatorname{Re} (\Xi_k) \\ &\quad + (\gamma - \lambda_k) 2 \operatorname{Re} \left(\overline{\Xi_k} \left[\cosh(\overline{z_k} T) \cosh(z_k T) - \frac{\overline{z_k}}{z_k} \sinh(\overline{z_k} T) \sinh(z_k T) \right] \right) \\ &= -2 \operatorname{Re} (\Xi_k^2) + 2 \operatorname{Re} \left(\overline{\Xi_k} \Xi_k \cosh(\overline{z_k} T) \cosh(z_k T) - \frac{\overline{z_k}}{z_k} \overline{\Xi_k} \Xi_k \sinh(\overline{z_k} T) \sinh(z_k T) \right) \\ &= -2 \operatorname{Re} (\Xi_k^2) + 2 |\Xi_k|^2 \operatorname{Re} \left(\cosh(\overline{z_k} T) \cosh(z_k T) - \frac{\overline{z_k}}{z_k} \sinh(\overline{z_k} T) \sinh(z_k T) \right) \\ &= d(k, T). \end{aligned}$$

due to the definition (5.25). Since $d(k, T) \neq 0$, this yields

$$-G''_{k,T}(0) + (\gamma - \lambda_k) G_{k,T}(0) = 1.$$

Verification of the Basis Function $H_{k,T}(t)$. Now we consider $H_{k,T}(t)$ from (5.23) that is multiplied with $(1 + \gamma \lambda_k) a'_{k,T}(0)$, i.e.

$$\begin{aligned} & d(k, T) H_{k,T}(t) \\ &= 2 \operatorname{Re} \left(\frac{\overline{\Xi_k}}{z_k} \sinh(\overline{z_k} T) \cosh(z_k(t - T)) + \frac{\overline{\Xi_k}}{z_k} \cosh(\overline{z_k} T) \sinh(z_k(t - T)) \right) \\ & \quad - 2 \operatorname{Re} \left(\frac{\Xi_k}{z_k} [\sinh(z_k T) \cosh(z_k(t - T)) + \cosh(z_k T) \sinh(z_k(t - T))] \right). \end{aligned}$$

Due to the definition of z_k , the function $H_{k,T}(t)$ satisfies the ODE (5.20). Moreover, for $t = T$ we have

$$d(k, T) H_{k,T}(T) = 2 \operatorname{Re} \left(\frac{\overline{\Xi_k}}{z_k} \sinh(\overline{z_k} T) \right) - 2 \operatorname{Re} \left(\frac{\Xi_k}{z_k} \sinh(z_k T) \right) = 0.$$

For the derivative we have

$$\begin{aligned} d(k, T) H'_{k,T}(t) &= 2 \operatorname{Re} \left(\overline{\Xi_k} \frac{z_k}{z_k} \sinh(\overline{z_k} T) \sinh(z_k(t - T)) + \overline{\Xi_k} \cosh(\overline{z_k} T) \cosh(z_k(t - T)) \right) \\ & \quad - 2 \operatorname{Re} (\Xi_k [\sinh(z_k T) \sinh(z_k(t - T)) + \cosh(z_k T) \cosh(z_k(t - T))]). \end{aligned}$$

Hence for $t = T$ we obtain $H'_{k,T}(T) = 0$.

For the second derivative, we have

$$\begin{aligned} d(k, T) H''_{k,T}(t) &= 2 \operatorname{Re} \left(\overline{\Xi_k} \frac{z_k^2}{z_k} \sinh(\overline{z_k} T) \cosh(z_k(t - T)) + \overline{\Xi_k} z_k \cosh(\overline{z_k} T) \sinh(z_k(t - T)) \right) \\ & \quad - 2 \operatorname{Re} (\Xi_k z_k [\sinh(z_k T) \cosh(z_k(t - T)) + \cosh(z_k T) \sinh(z_k(t - T))]). \end{aligned}$$

For $t = 0$ this implies

$$d(k, T) H''_{k,T}(0) = 2 \operatorname{Re} \left(\overline{\Xi_k} \frac{z_k^2}{z_k} \sinh(\overline{z_k} T) \cosh(z_k T) - \overline{\Xi_k} z_k \cosh(\overline{z_k} T) \sinh(z_k T) \right).$$

We have

$$d(k, T) H_{k,T}(0) = 2 \operatorname{Re} \left(\frac{\overline{\Xi_k}}{z_k} \sinh(\overline{z_k} T) \cosh(z_k T) - \frac{\overline{\Xi_k}}{z_k} \cosh(\overline{z_k} T) \sinh(z_k T) \right).$$

Hence in view of (5.24) we obtain

$$\begin{aligned} & -d(k, T) H''_{k,T}(0) + (\gamma - \lambda_k) \det(M(T)) H_{k,T}(0) \\ &= 2 \operatorname{Re} \left(-\overline{\Xi_k} \frac{z_k^2}{z_k} \sinh(\overline{z_k} T) \cosh(z_k T) + \overline{\Xi_k} z_k \cosh(\overline{z_k} T) \sinh(z_k T) \right) \\ & \quad + (\gamma - \lambda_k) 2 \operatorname{Re} \left(\frac{\overline{\Xi_k}}{z_k} \sinh(\overline{z_k} T) \cosh(z_k T) - \frac{\overline{\Xi_k}}{z_k} \cosh(\overline{z_k} T) \sinh(z_k T) \right) \\ &= 2 \operatorname{Re} \left(\overline{\Xi_k} \Xi_k \frac{1}{z_k} \sinh(\overline{z_k} T) \cosh(z_k T) - \Xi_k \frac{\overline{\Xi_k}}{z_k} \cosh(\overline{z_k} T) \sinh(z_k T) \right) = 0. \end{aligned}$$

Now we verify that the fourth boundary condition in (5.22) is satisfied. We have

$$d(k, T) H'_{k,T}(0) = 2 \operatorname{Re} \left(-\overline{\Xi_k} \frac{z_k}{\overline{z_k}} \sinh(\overline{z_k} T) \sinh(z_k T) + \overline{\Xi_k} \cosh(\overline{z_k} T) \cosh(z_k T) \right) - 2 \operatorname{Re}(\Xi_k).$$

For the third derivative, we have

$$\begin{aligned} & d(k, T) H'''_{k,T}(t) \\ &= 2 \operatorname{Re} \left(\overline{\Xi_k} \frac{z_k^3}{\overline{z_k}} \sinh(\overline{z_k} T) \sinh(z_k(t - T)) + \overline{\Xi_k} z_k^2 \cosh(\overline{z_k} T) \cosh(z_k(t - T)) \right) \\ & - 2 \operatorname{Re} \left(\Xi_k z_k^2 [\sinh(z_k T) \sinh(z_k(t - T)) + \cosh(z_k T) \cosh(z_k(t - T))] \right). \end{aligned}$$

For $t = 0$ this yields

$$\begin{aligned} & d(k, T) H'''_{k,T}(0) \\ &= 2 \operatorname{Re} \left(-\overline{\Xi_k} \frac{z_k^3}{\overline{z_k}} \sinh(\overline{z_k} T) \sinh(z_k T) + \overline{\Xi_k} z_k^2 \cosh(\overline{z_k} T) \cosh(z_k T) \right) \\ & - 2 \operatorname{Re} \left(\Xi_k z_k^2 [-\sinh(z_k T) \sinh(z_k T) + \cosh(z_k T) \cosh(z_k T)] \right) \\ &= 2 \operatorname{Re} \left(-\overline{\Xi_k} \frac{z_k^3}{\overline{z_k}} \sinh(\overline{z_k} T) \sinh(z_k T) + \overline{\Xi_k} z_k^2 \cosh(\overline{z_k} T) \cosh(z_k T) \right) \\ & - 2 \operatorname{Re}(\Xi_k z_k^2). \end{aligned}$$

Using again the definition (5.24) of Ξ_k it follows that

$$\begin{aligned} & -d(k, T) H'''_{k,T}(0) + (\gamma - \lambda_k) d(k, T) H'_{k,T}(0) \\ &= 2 \operatorname{Re} \left(\overline{\Xi_k} \frac{z_k^3}{\overline{z_k}} \sinh(\overline{z_k} T) \sinh(z_k T) - \overline{\Xi_k} z_k^2 \cosh(\overline{z_k} T) \cosh(z_k T) \right) + 2 \operatorname{Re}(\Xi_k z_k^2) \\ & + (\gamma - \lambda_k) 2 \operatorname{Re} \left(-\overline{\Xi_k} \frac{z_k}{\overline{z_k}} \sinh(\overline{z_k} T) \sinh(z_k T) + \overline{\Xi_k} \cosh(\overline{z_k} T) \cosh(z_k T) \right) \\ & - (\gamma - \lambda_k) 2 \operatorname{Re}(\Xi_k) \\ &= -2 \operatorname{Re}(\Xi_k^2) + 2 \operatorname{Re} \left(-\Xi_k \overline{\Xi_k} \frac{z_k}{\overline{z_k}} \sinh(\overline{z_k} T) \sinh(z_k T) + \Xi_k \overline{\Xi_k} \cosh(\overline{z_k} T) \cosh(z_k T) \right) \\ &= -2 \operatorname{Re}(\Xi_k^2) + |\Xi_k|^2 2 \operatorname{Re} \left(\cosh(\overline{z_k} T) \cosh(z_k T) - \frac{z_k}{\overline{z_k}} \sinh(\overline{z_k} T) \sinh(z_k T) \right) \\ &= d(k, T) \end{aligned}$$

by the definition (5.25) of $d(k, T)$. Thus we have shown that

$$\begin{aligned} H_{k,T}(T) &= 0, \\ H'_{k,T}(T) &= 0, \\ -H''_{k,T}(0) + (\gamma - \lambda_k) H_{k,T}(0) &= 0, \end{aligned}$$

$$-H_{k,T}'''(0) + (\gamma - \lambda_k) H_{k,T}'(0) = 1.$$

The proof of Lemma 2 is complete.

In the next section, it is shown that each of the functions $F_{k,T}$, $G_{k,T}$, $H_{k,T}$ in the representation (5.23) of the solution $b_{k,T}$ that are provided in Lemma 2 satisfies an exponential turnpike inequality on the interval $[0, T]$ in the sense that for $I \in \{F, G, H\}$ for all $t \in [0, T]$ the following inequality holds:

$$|I_{k,T}(t)| \leq C_{\textcircled{a}} \left[\exp\left(-\frac{\sqrt{\gamma}}{2}t\right) + \exp\left(-\frac{\sqrt{\gamma}}{2}(T-t)\right) \right].$$

Here $C_{\textcircled{a}}$ is a constant that is independent of T and k .

The turnpike inequality that we have obtained is used for the applications in shape optimization as described in the subsequent sections. An important point is that the inequality is independent of the properties of the sequence of eigenvalues λ_k as long as (5.18) holds. Therefore the inequality is valid over a compact set of perturbations of the state equation (see Theorem 10 and also Remark 7).

5.4 The Turnpike Property by the Spectral Method for trees

In this section, we continue our analysis of the structure of the optimal solutions, in particular for the adjoint states. We have in mind the shape optimization problems for the trees. Therefore, we restrict our analysis to the networks in the form of trees. The case of small cycles for the purposes of topology optimization is considered separately. In the latter case, the spectrum is of a specific structure with the branch supported exclusively on the cycle.

We show that the difference between the static optimal adjoint state and the dynamic optimal adjoint state satisfies an exponential turnpike inequality as explained above. Our method is to show that all three basis functions in the representation of the difference between the static optimal adjoint state and the dynamic optimal adjoint state that we have obtained from the respective optimality systems satisfy such an exponential turnpike inequality. More precisely, we show that the basis function with a non-zero value at the time $t = 0$ (namely $G_{k,T}$ and $H_{k,T}$) decay exponentially fast with t . The basis function with a non-zero value at the time $t = T$ (namely $F_{k,T}$) decay exponentially fast as a function of $T - t$, that is backward in time.

5.4.1 The Turnpike Inequalities for the Basis Functions

Exponential Turnpike Inequality for the Basis Function $F_{k,T}(t)$ In order to verify the turnpike property we use the following representation where the hyperbolic tangent appears:

$$\begin{aligned}
& d(k, T) F_{k,T}(t) \\
&= -2 \operatorname{Re} \left(\overline{\Xi_k}^2 \cosh(z_k(T-t)) \right) \\
&+ |\Xi_k|^2 2 \operatorname{Re} \left(\cosh(z_k T) \cosh(\overline{z_k} T) [1 - \tanh(z_k T) \tanh(z_k(T-t))] \cosh(z_k(T-t)) \right) \\
&+ |\Xi_k|^2 2 \operatorname{Re} \left(\cosh(z_k T) \cosh(\overline{z_k} T) \frac{\overline{z_k}}{z_k} [\tanh(z_k(T-t)) - \tanh(z_k T)] \tanh(\overline{z_k} T) \cosh(z_k(T-t)) \right).
\end{aligned}$$

For our analysis, we define the auxiliary functions

$$\begin{aligned}
S_{k,T}^1(t) &= \frac{-2 \operatorname{Re} \left(\overline{\Xi_k}^2 \cosh(z_k(T-t)) \right)}{d(k, T)}, \\
S_{k,T}^2(t) &= \frac{|\Xi_k|^2 2 \operatorname{Re} \left(\cosh(z_k T) \cosh(\overline{z_k} T) [1 - \tanh(z_k T) \tanh(z_k(T-t))] \cosh(z_k(T-t)) \right)}{d(k, T)}, \\
&\text{and} \\
S_{k,T}^3(t) &= \frac{|\Xi_k|^2 2 \operatorname{Re} \left(\cosh(z_k T) \cosh(\overline{z_k} T) \frac{\overline{z_k}}{z_k} [\tanh(z_k(T-t)) - \tanh(z_k T)] \tanh(\overline{z_k} T) \cosh(z_k(T-t)) \right)}{d(k, T)}.
\end{aligned}$$

Then we have

$$F_{k,T}(t) = S_{k,T}^1(t) + S_{k,T}^2(t) + S_{k,T}^3(t). \quad (5.29)$$

The numbers $d(k, T)$ can be represented as

$$\begin{aligned}
& d(k, T) \\
&= -2 \operatorname{Re} (\Xi_k^2) + |\Xi_k|^2 \left[\left(1 - \frac{\frac{\gamma}{2} - \lambda_k}{\sqrt{1 + \lambda_k^2}} \right) \cosh(2 \operatorname{Re}(z_k) T) + \left(1 + \frac{\frac{\gamma}{2} - \lambda_k}{\sqrt{1 + \lambda_k^2}} \right) \cos(2 \operatorname{Im}(z_k) T) \right].
\end{aligned}$$

Hence we obtain the upper bound

$$|S_{k,T}^1(t)| \leq \frac{2 |\cosh(z_k(T-t))|}{\left(1 - \frac{\frac{\gamma}{2} - \lambda_k}{\sqrt{1 + \lambda_k^2}} \right) \cosh(2 \operatorname{Re}(z_k) T) - \left| \left(1 + \frac{\frac{\gamma}{2} - \lambda_k}{\sqrt{1 + \lambda_k^2}} \right) \cos(2 \operatorname{Im}(z_k) T) - 2 \operatorname{Re} \left(\frac{\Xi_k^2}{|\Xi_k|^2} \right) \right|}.$$

This implies the inequality

$$|S_{k,T}^1(t)| \leq M_1 \exp(-|\operatorname{Re}(z_k)|(T-t)) \quad (5.30)$$

with a constant $M_1 \geq 1$ that is independent of k and T . We have

$$1 - \tanh(z_k T) \tanh(z_k(T-t)) = \frac{2[\exp(2 z_k T) + \exp(2 z_k(T-t))]}{(\exp(2 z_k T) + 1)(\exp(2 z_k(T-t)) + 1)}$$

$$= \frac{2}{(1 + \exp(-2 z_k T)) (\exp(2 z_k (T - t)) + 1)} + \frac{2}{(1 + \exp(-2 z_k (T - t))) (\exp(2 z_k T) + 1)}.$$

Hence for $T \rightarrow \infty$ this term converges to zero exponentially fast. More precisely, due to (5.21) for all $k \in \{0, 1, 2, \dots\}$ for T sufficiently large we have the inequality

$$|1 - \tanh(z_k T) \tanh(z_k (T - t))| \leq 16 \exp(-2 |\operatorname{Re}(z_k)| (T - t)). \quad (5.31)$$

Thus we have

$$\begin{aligned} & |S_{k,T}^2(t)| \\ & \leq \frac{32 |\operatorname{Re}(\cosh(z_k T) \cosh(\bar{z}_k T) \cosh(z_k (T - t)))| \exp(-2 |\operatorname{Re}(z_k)| (T - t))}{\left| \left(1 - \frac{\frac{\gamma}{2} - \lambda_k}{\sqrt{1 + \lambda_k^2}}\right) \cosh(2 \operatorname{Re}(z_k) T) - \left(1 + \frac{\frac{\gamma}{2} - \lambda_k}{\sqrt{1 + \lambda_k^2}}\right) \cos(2 \operatorname{Im}(z_k) T) - 2 \operatorname{Re}\left(\frac{\Xi_k^2}{|\Xi_k|^2}\right) \right|}. \end{aligned}$$

This implies the inequality

$$|S_{k,T}^2(t)| \leq M_2 \exp(-|\operatorname{Re}(z_k)| (T - t)) \quad (5.32)$$

with a constant $M_2 \geq 1$ that is independent of k and T .

In addition, we have

$$\tanh(z_k (T - t)) - \tanh(z_k T) = \frac{2[\exp(2 z_k T) - \exp(2 z_k (T - t))]}{(\exp(2 z_k T) + 1) (\exp(2 z_k (T - t)) + 1)}$$

so as above for T sufficiently large we obtain the inequality

$$|\tanh(z_k (T - t)) - \tanh(z_k T)| \leq 16 \exp(-2 |\operatorname{Re}(z_k)| (T - t)). \quad (5.33)$$

This yields the bound

$$\begin{aligned} & |S_{k,T}^3(t)| \\ & \leq \frac{32 |\operatorname{Re}(\cosh(z_k T) \cosh(\bar{z}_k T) \cosh(z_k (T - t)) \tanh(\bar{z}_k T))| \exp(-2 |\operatorname{Re}(z_k)| (T - t))}{\left| \left(1 - \frac{\frac{\gamma}{2} - \lambda_k}{\sqrt{1 + \lambda_k^2}}\right) \cosh(2 \operatorname{Re}(z_k) T) - \left(1 + \frac{\frac{\gamma}{2} - \lambda_k}{\sqrt{1 + \lambda_k^2}}\right) \cos(2 \operatorname{Im}(z_k) T) - 2 \operatorname{Re}\left(\frac{\Xi_k^2}{|\Xi_k|^2}\right) \right|}. \end{aligned}$$

Thus we obtain the inequality

$$|S_{k,T}^3(t)| \leq M_3 \exp(-|\operatorname{Re}(z_k)| (T - t))$$

with a constant $M_3 \geq 1$ that is independent of k and T . Together with (5.30) and (5.32) due to (5.29) and (5.21) this implies that for $F_{k,T}(t)$ we have

$$|F_{k,T}(t)| \leq (M_1 + M_2 + M_3) \exp\left(-\frac{\sqrt{\gamma}}{2}(T - t)\right). \quad (5.34)$$

Exponential Turnpike Inequality for the Basis Function $G_{k,T}(t)$ Now we derive the turnpike inequality for $G_{k,T}(t)$. Again we use a representation with the hyperbolic tangent. We have

$$\begin{aligned} d(k, T)G_{k,T}(t) &= -2\operatorname{Re}(\Xi_k \cosh(z_k T) [1 + \tanh(z_k T) \tanh(z_k(t - T))] \cosh(z_k(t - T))) \\ &\quad + 2\operatorname{Re}\left(\overline{\Xi_k} \cosh(\overline{z_k} T) \left[1 + \frac{\overline{z_k}}{z_k} \tanh(\overline{z_k} T) \tanh(z_k(t - T))\right] \cosh(z_k(t - T))\right). \end{aligned}$$

We have

$$\lim_{T \rightarrow \infty} \left[1 + \frac{\overline{z_k}}{z_k} \tanh(\overline{z_k} T) \tanh(z_k(t - T))\right] = 1 - \frac{\overline{z_k}}{z_k}.$$

Similarly, we have

$$\lim_{T \rightarrow \infty} [1 + \tanh(z_k T) \tanh(z_k(t - T))] = 0$$

and for the absolute value we have the bound (5.31).

For our analysis, we define the auxiliary functions

$$\begin{aligned} \tilde{S}_{k,T}^1(t) &= \frac{-2 \operatorname{Re}(\Xi_k \cosh(z_k T) [1 + \tanh(z_k T) \tanh(z_k(t - T))] \cosh(z_k(t - T)))}{d(k, T)}, \\ \tilde{S}_{k,T}^2(t) &= \frac{2 \operatorname{Re}\left(\overline{\Xi_k} \cosh(\overline{z_k} T) \left[1 + \frac{\overline{z_k}}{z_k} \tanh(\overline{z_k} T) \tanh(z_k(t - T))\right] \cosh(z_k(t - T))\right)}{d(k, T)}. \end{aligned}$$

Then we have

$$G_{k,T}(t) = \tilde{S}_{k,T}^1(t) + \tilde{S}_{k,T}^2(t). \quad (5.35)$$

Due to (5.31) we obtain the inequality

$$\begin{aligned} &|\tilde{S}_{k,T}^1(t)| \\ &\leq \frac{32 \left| \operatorname{Re}\left(\frac{\Xi_k}{|\Xi_k|^2} \cosh(z_k T) \cosh(z_k(t - T))\right) \right| \exp(-2 |\operatorname{Re}(z_k)| (T - t))}{\left(1 - \frac{\frac{\gamma}{2} - \lambda_k}{\sqrt{1 + \lambda_k^2}}\right) \cosh(2 \operatorname{Re}(z_k) T) - \left|\left(1 + \frac{\frac{\gamma}{2} - \lambda_k}{\sqrt{1 + \lambda_k^2}}\right) \cos(2 \operatorname{Im}(z_k) T) - 2 \operatorname{Re}\left(\frac{\Xi_k^2}{|\Xi_k|^2}\right)\right|}. \end{aligned}$$

This implies the inequality

$$|\tilde{S}_{k,T}^1(t)| \leq \frac{1}{|\Xi_k|} \tilde{M}_1 \exp(-|\operatorname{Re}(z_k)| (T - t)) \quad (5.36)$$

with a constant $\tilde{M}_1 > 0$ that is independent of k and T . Moreover, we have

$$\begin{aligned} &|\tilde{S}_{k,T}^2(t)| \\ &\leq \frac{32 \left| \operatorname{Re}\left(\frac{\overline{\Xi_k}}{|\Xi_k|^2} \cosh(\overline{z_k} T) \cosh(z_k(t - T))\right) \right|}{\left(1 - \frac{\frac{\gamma}{2} - \lambda_k}{\sqrt{1 + \lambda_k^2}}\right) \cosh(2 \operatorname{Re}(z_k) T) - \left|\left(1 + \frac{\frac{\gamma}{2} - \lambda_k}{\sqrt{1 + \lambda_k^2}}\right) \cos(2 \operatorname{Im}(z_k) T) - 2 \operatorname{Re}\left(\frac{\Xi_k^2}{|\Xi_k|^2}\right)\right|}. \end{aligned}$$

This implies the inequality

$$|\tilde{S}_{k,T}^2(t)| \leq \frac{1}{|\Xi_k|} \tilde{M}_2 [\exp(-|\operatorname{Re}(z_k)|t) + \exp(-|\operatorname{Re}(z_k)|(T-t))] \quad (5.37)$$

with a constant $\tilde{M}_2 > 0$ that is independent of k and T .

With (5.36) and (5.37) due to (5.35) and (5.21) this implies that for $G_{k,T}(t)$ we have

$$|G_{k,T}(t)| \leq \frac{1}{|\Xi_k|} (\tilde{M}_1 + \tilde{M}_2) \left[\exp(-\frac{\sqrt{\gamma}}{2}t) + \exp(-\frac{\sqrt{\gamma}}{2}(T-t)) \right]. \quad (5.38)$$

Exponential Turnpike Inequality for the Basis Function $H_{k,T}(t)$ Finally, we show the exponential turnpike inequality for the third basis function.

For our analysis, we define the auxiliary functions

$$\hat{S}_{k,T}^1(t) = \frac{2 \operatorname{Re} \left(\cosh(\overline{z_k}T) \left[\frac{\overline{\Xi_k}}{z_k} \tanh(\overline{z_k}T) + \frac{\overline{\Xi_k}}{z_k} \tanh(z_k(t-T)) \right] \cosh(z_k(t-T)) \right)}{d(k, T)}$$

and

$$\hat{S}_{k,T}^2(t) = \frac{2 \operatorname{Re} \left(\cosh(z_kT) \frac{\Xi_k}{z_k} [\tanh(z_kT) + \tanh(z_k(t-T))] \cosh(z_k(t-T)) \right)}{d(k, T)}.$$

Then we have

$$H_{k,T}(t) = \hat{S}_{k,T}^1(t) - \hat{S}_{k,T}^2(t). \quad (5.39)$$

For the factor with the hyperbolic tangent that appears in $\hat{S}_{k,T}^1(t)$ for T sufficiently large we have the bound

$$\begin{aligned} & \left| \frac{\overline{\Xi_k}}{z_k} \tanh(\overline{z_k}T) + \frac{\overline{\Xi_k}}{z_k} \tanh(z_k(t-T)) \right| \\ & \leq \left| \frac{\Xi_k}{z_k} \right| [|\tanh(\overline{z_k}T)| + |\tanh(z_k(t-T))|] \leq 4\sqrt{\gamma + \frac{1}{\lambda_k}}. \end{aligned}$$

This yields the inequality

$$\begin{aligned} & |\hat{S}_{k,T}^1(t)| \\ & \leq \frac{\frac{8}{|\Xi_k|^2} \left| \operatorname{Re} \left(\cosh(\overline{z_k}T) \sqrt{\gamma + \frac{1}{\lambda_k}} \cosh(z_k(t-T)) \right) \right|}{\left(1 - \frac{\frac{\gamma - \lambda_k}{2}}{\sqrt{1 + \lambda_k^2}} \right) \cosh(2 \operatorname{Re}(z_k) T) - \left| \left(1 + \frac{\frac{\gamma - \lambda_k}{2}}{\sqrt{1 + \lambda_k^2}} \right) \cos(2 \operatorname{Im}(z_k) T) - 2 \operatorname{Re} \left(\frac{\Xi_k^2}{|\Xi_k|^2} \right) \right|}. \end{aligned}$$

This implies the inequality

$$|\hat{S}_{k,T}^1(t)| \leq \frac{1}{|\Xi_k|^2} \hat{M}_1 [\exp(-|\operatorname{Re}(z_k)|t) + \exp(-|\operatorname{Re}(z_k)|(T-t))] \quad (5.40)$$

with a constant $\hat{M}_1 > 0$ that is independent of k and T .

For the factor with the hyperbolic tangent that appears in $\hat{S}_{k,T}^2(t)$ for T sufficiently large we have the bound

$$|\tanh(z_k T) + \tanh(z_k(t - T))| \leq 4.$$

This yields the inequality

$$\begin{aligned} & |\hat{S}_{k,T}^2(t)| \\ & \leq \frac{\frac{8}{|\Xi_k|^2} \left| \operatorname{Re} \left(\cosh(z_k T) \sqrt{\gamma + \frac{1}{\lambda_k}} \cosh(z_k(t - T)) \right) \right|}{\left(1 - \frac{\frac{\gamma}{2} - \lambda_k}{\sqrt{1 + \lambda_k^2}} \right) \cosh(2 \operatorname{Re}(z_k) T) - \left| \left(1 + \frac{\frac{\gamma}{2} - \lambda_k}{\sqrt{1 + \lambda_k^2}} \right) \cos(2 \operatorname{Im}(z_k) T) - 2 \operatorname{Re} \left(\frac{\Xi_k^2}{|\Xi_k|^2} \right) \right|}. \end{aligned}$$

This implies the inequality

$$|\hat{S}_{k,T}^2(t)| \leq \frac{1}{|\Xi_k|^2} \hat{M}_2 [\exp(-|\operatorname{Re}(z_k)| t) + \exp(-|\operatorname{Re}(z_k)|(T - t))] \quad (5.41)$$

with a constant $\hat{M}_2 > 0$ that is independent of k and T . With (5.40) and (5.41) due to (5.39) and (5.21) this implies that for $H_{k,T}(t)$ we have

$$|H_{k,T}(t)| \leq \frac{1}{|\Xi_k|^2} (\hat{M}_1 + \hat{M}_2) \left[\exp(-\frac{\sqrt{\gamma}}{2} t) + \exp(-\frac{\sqrt{\gamma}}{2} (T - t)) \right]. \quad (5.42)$$

5.4.2 Theorem on Turnpike Property Between the Dynamic and Static Optimality Systems

We have unique solutions for the optimality systems governed by static and dynamic state equations. Now, we compare the elements of optimality systems and obtain the turnpike inequalities for the Optimality Systems.

Indeed, the turnpike inequalities that we have derived lead to the useful for applications Turnpike Theorem which means that :

The difference ν^T of the optimal dynamic control and the optimal static control and the corresponding differences ω^T for the state and μ^T the adjoint state admits an exponential turnpike property.

To this end, we need some additional regularity assumptions.

Theorem 10. *Assume that (5.18) holds and that the initial state satisfies the regularity condition*

$$\sum_{k=0}^{\infty} \lambda_k |a_k(0)|^2 + |a'_k(0)|^2 < \infty, \quad (5.43)$$

that is the initial state belongs to the energy space of the elliptic problem defined by the bilinear form $a(\cdot, \cdot)$. If $\Omega = \Gamma$ then there exists a constant $\tilde{D} = \tilde{D}(y_0, y_1, p^\sigma)$ that is independent of T and t such that for all $t \in [0, T]$

$$\|\omega^T(t)\|_{L^2(\Omega)}^2 + \|\nu^T(t)\|_{L^2(\Omega)}^2 + \|\mu^T(t)\|_{L^2(\Omega)}^2 \leq \tilde{D} [e^{-\sqrt{\gamma}t} + e^{-\sqrt{\gamma}(T-t)}]. \quad (5.44)$$

Moreover, the constant $\tilde{D}$ depends on Ω only as a function of the energy norm for the initial state that is determined by Ω as in (5.43).

Remark 7. In the case of boundary control problems on networks, the Turnpike Property for optimality systems can be shown. The difference between the boundary control and the distributed control is the appearance of a linear operator in the optimality conditions. In the optimality conditions of the boundary control problem, a linear operator maps the adjoint state into the optimal control.

For the optimal cost the exponential turnpike inequality (5.44) implies the so-called integral turnpike property (see e.g. [23])

$$\sup_{T>0} \int_0^T \|\omega^T(t)\|_{L^2(\Omega)}^2 + \|\nu^T(t)\|_{L^2(\Omega)}^2 + \|\mu^T(t)\|_{L^2(\Omega)}^2 dt < \infty.$$

This implies in turn

$$\lim_{T \rightarrow \infty} \frac{1}{T} J_T(\hat{u}^T) = I(\hat{v}), \quad (5.45)$$

see for example [26].

In shape sensitivity analysis [80] we avoid, if possible, the dependence of the shape gradient of the cost with respect to given data including e.g., the given initial conditions. Thus, we define the initial conditions, say $y_0(x)$ and $y_1(x)$, $x \in \Omega_\tau$, for variable domains $\tau \rightarrow \Omega_\tau$ used for the derivation of shape gradient of the cost. If Ω_τ is e.g., the cross with variable length of the edges, we select the elements y_0, y_1 as the restrictions to Ω_τ of functions $Y_0(x), Y_1(x)$ defined on the cross with the edges of maximal lengths. The shape derivatives of the restrictions are zero. Therefore, there is no contribution from the initial conditions to the shape gradient of the cost.

Remark 8. We use the material derivative method [80] for the purposes of shape sensitivity analysis for networks. The general rule for the data of initial-boundary value problems is to select the given functions by the restriction to actual domain of some functions defined everywhere. The shape gradients of such initial data are simply zero and the material derivatives are given by the gradients, thus some regularity is required. In this way the shape gradient of the cost is independent of the initial data. In our case, this selection can be used for the initial conditions of the displacement and the velocity. The material derivatives of such initial conditions $Y = Y(x)$ take the form $Y'(x)\mathcal{V}(0, x)$, where $\mathcal{V}(\tau, x)$ is the velocity field of the material derivative method. In other words, the initial conditions are selected in such a way that there is no contribution of the initial conditions to the shape gradient of the cost function.

The following corollary states a shape-turnpike result. It is a relation between the optimal values of a dynamic optimal shape problem for large time horizons and the optimal values for the static optimal shape problem. For the proof, we suggest to proceed by contradiction.

Assumption 1. Let us consider a tree $G = \{E, V\}$ with the set of edges $E_i = [0, L_i]$, $i = 1, \dots, N$, and denote by $\ell = \text{col}(L_1, \dots, L_N)$. The set of admissible trees $\Omega(\ell) \in \mathcal{A}$ is defined by the conditions $M_i^{\min} \leq L_i \leq M_i^{\max}$, where $0 < M_0 \leq M_i^{\min} < M_i^{\max} < M_1 < \infty$. The set of admissible trees is convex and compact, therefore, for the minimizing sequence ℓ_n of optimization problem, there is a subsequence, still denoted by the same symbol such that we have $\ell_n \rightarrow \ell_\infty$ in $\mathbb{R}^N$ and in addition $\Omega(\ell_\infty) \in \mathcal{A}$.

Corollary 1. Let Assumption 1 hold. Let a sequence of shape parameters $(\ell_n)_n$ and a bounded sequence of controls $(\hat{u}_n)_n$ with $\hat{u}_n(t) \in L^2(\Omega(\ell_n))$ for all $n \in \{1, 2, 3, \dots\}$ be given. Let $\hat{y}_n$ denote the generated state and $\hat{p}_n$ the corresponding adjoint state. Assume that for all $n \in \{1, 2, 3, \dots\}$ we have

$$\|\omega_n^T(t)\|_{L^2(\Omega(\ell_n))}^2 + \|\nu_n^T(t)\|_{L^2(\Omega(\ell_n))}^2 + \|\mu_n^T(t)\|_{L^2(\Omega(\ell_n))}^2 \leq \tilde{D} [e^{-\sqrt{\gamma}t} + e^{-\sqrt{\gamma}(T-t)}] \quad (5.46)$$

where $\omega_n^T = \hat{y}_n^T - \hat{z}_n^\sigma$, $\mu_n^T = \hat{p}_n^T - \hat{p}_n^\sigma$, $\nu_n^T = \hat{u}_n^T - \hat{u}_n^\sigma$ and $(\hat{u}_n, \hat{z}_n, \hat{p}_n)$ is optimal for $\Omega(\ell_n)$

Assume that $\lim_{n \rightarrow \infty} \ell_{\text{opt}}(T_n) = \hat{\ell}$ and that $\hat{u}_n$ converges weakly to u_{opt} .

Assume that the optimal shape problem with (OCE) has a solution $(\ell_{\text{opt}}, u_{\text{opt}})$. Then u_{opt} is a solution of (OCE).

Let $\hat{v}_{\text{opt}}$ denote the solution of (OCS) for ℓ_{opt} .

Then we have

$$\|\omega_{\text{opt}}^T(t)\|_{L^2(\Omega(\ell_{\text{opt}}))}^2 + \|\nu_{\text{opt}}^T(t)\|_{L^2(\Omega(\ell_{\text{opt}}))}^2 + \|\mu_{\text{opt}}^T(t)\|_{L^2(\Omega(\ell_{\text{opt}}))}^2 \leq \tilde{D} [e^{-\sqrt{\gamma}t} + e^{-\sqrt{\gamma}(T-t)}]. \quad (5.47)$$

Assume that for a subsequence we have $\lim_{T_n \rightarrow \infty} \ell_{\text{opt}}(T_n) = \hat{\ell}$.

Then we have

$$\lim_{n \rightarrow \infty} \frac{1}{T_n} J_{T_n}(\hat{u}_{\text{opt}}^{T_n}, \ell_{\text{opt}}(T_n)) = I(\hat{v}, \hat{\ell}). \quad (5.48)$$

and $\hat{v}$ is the optimal control for the network defined in $\Omega(\hat{\ell})$, the limit shape parameter $\hat{\ell}$ is optimal for the static problem $\ell \rightarrow (\text{OCS})(\ell)$, the optimal shape reads $\Omega(\hat{\ell})$. In general, an optimal shape is not unique but it does exist for the compact set of admissible shapes.

Remark 9. Under Assumption 1 the set of admissible shapes is convex and compact in $\mathbb{R}^N$. Denote by $\hat{\ell}$ an optimal shape for the static problem, note that the optimal shape is not unique, and let the admissible sequence of shapes ℓ_n be convergent to an optimal shape for $n \rightarrow \infty$,

$$\ell_n \rightarrow \hat{\ell}$$

then

$$I(\hat{v}, \hat{\ell}) = \lim_{n \rightarrow \infty} \frac{1}{T_n} J_{T_n}(\hat{u}_{\text{opt}}^{T_n}, \ell_n) \geq \lim_{n \rightarrow \infty} \frac{1}{T_n} J_{T_n}(\hat{u}_{\text{opt}}^{T_n}, \ell_{\text{opt}}(T_n)) = I(\hat{v}, \tilde{\ell}) \quad (5.49)$$

therefore

$$I(\hat{v}, \hat{\ell}) = I(\hat{v}, \tilde{\ell}).$$

Proof of the corollary.

The constant control with the value $\hat{v}$ can be considered as an element of $L^2(\Omega(\ell_{opt}(T_n)))$. We have the inequality

$$\begin{aligned} & \left| \|\hat{u}_{opt}^{T_n}(\ell_{opt}(T_n)) - u_d\|_{L^2(\Omega(\ell_{opt}(T_n)))}^2 - \|\hat{v} - u_d\|_{L^2(\Omega(\ell_{opt}(T_n)))}^2 \right| \\ & \leq \|\hat{u}_{opt}^{T_n}(\ell_{opt}(T_n)) - \hat{v}\|_{L^2(\Omega(\ell_{opt}(T_n)))} \left(\|\hat{u}_{opt}^{T_n}(\ell_{opt}(T_n)) - u_d\|_{L^2(\Omega(\ell_{opt}(T_n)))} + \|\hat{v} - u_d\|_{L^2(\Omega(\ell_{opt}(T_n)))} \right). \end{aligned}$$

Integration from 0 to T_n and division by T_n yields

$$\begin{aligned} & \left| \frac{\int_0^{T_n} \|\hat{u}_{opt}^{T_n}(\ell_{opt}(T_n)) - u_d\|_{L^2(\Omega(\ell_{opt}(T_n)))}^2 dt}{T_n} - \|\hat{v} - u_d\|_{L^2(\Omega(\ell_{opt}(T_n)))}^2 \right| \\ & \leq \frac{\int_0^{T_n} \|\hat{u}_{opt}^{T_n}(\ell_{opt}(T_n)) - \hat{v}\|_{L^2(\Omega(\ell_{opt}(T_n)))} \left(\|\hat{u}_{opt}^{T_n}(\ell_{opt}(T_n)) - u_d\|_{L^2(\Omega(\ell_{opt}(T_n)))} + \|\hat{v} - u_d\|_{L^2(\Omega(\ell_{opt}(T_n)))} \right) dt}{T_n} \\ & \leq \frac{\int_0^{T_n} \|\hat{u}_{opt}^{T_n}(\ell_{opt}(T_n)) - \hat{v}\|_{L^2(\Omega(\ell_{opt}(T_n)))}^2 dt}{T_n} \\ & \quad + \frac{2\|\hat{v} - u_d\|_{L^2(\Omega(\ell_{opt}(T_n)))} \int_0^{T_n} \|\hat{u}_{opt}^{T_n}(\ell_{opt}(T_n)) - \hat{v}\|_{L^2(\Omega(\ell_{opt}(T_n)))} dt}{T_n}. \end{aligned}$$

For the state we obtain a similar inequality, namely

$$\begin{aligned} & \left| \frac{\int_0^{T_n} \|\hat{g}_{opt}^{T_n}(\ell_{opt}(T_n)) - y_d\|_{L^2(\Omega(\ell_{opt}(T_n)))}^2 dt}{T_n} - \|\hat{z} - y_d\|_{L^2(\Omega(\ell_{opt}(T_n)))}^2 \right| \\ & \leq \frac{\int_0^{T_n} \|\hat{g}_{opt}^{T_n}(\ell_{opt}(T_n)) - \hat{z}\|_{L^2(\Omega(\ell_{opt}(T_n)))}^2 dt}{T_n} \\ & \quad + \frac{2\|\hat{z} - y_d\|_{L^2(\Omega(\ell_{opt}(T_n)))} \int_0^{T_n} \|\hat{g}_{opt}^{T_n}(\ell_{opt}(T_n)) - \hat{z}\|_{L^2(\Omega(\ell_{opt}(T_n)))} dt}{T_n}. \end{aligned}$$

For the optimal cost the exponential turnpike inequality (5.44) implies the integral turnpike property (see e.g. [23])

$$\sup_{T>0} \sup_{\ell} \int_0^T \|\omega^T(t)\|_{L^2(\Omega)}^2 + \|\nu^T(t)\|_{L^2(\Omega)}^2 + \|\mu^T(t)\|_{L^2(\Omega)}^2 dt < \infty.$$

Moreover, we have

$$\sup_{T>0} \sup_{\ell} \int_0^T \|\omega^T(t)\|_{L^2(\Omega)} + \|\nu^T(t)\|_{L^2(\Omega)} + \|\mu^T(t)\|_{L^2(\Omega)} dt < \infty.$$

Thus adding up the inequalities for the control and the state and taking the limit for $T_n \rightarrow \infty$ yields (5.48).

Remark 10. For tree-shaped graphs often the system is exactly controllable in some finite time $t_{\min}$. Recently it was shown that if there is also control action at the interior nodes of the graph, exact controllability is also possible for a graph with cycles, see [5]. In this case, we can choose for all $n \in \{1, 2, 3, \dots\}$ a control function $u_{init}^{(n)}(t) \in L^2(0, t_{\min}; L^2(\Omega(\ell_{opt}(T_n))))$ that steers the system to the constant state $y(t_{\min}, \cdot) = \hat{y}$, $y_t(t_{\min}, \cdot) = 0$ and satisfies

$$\max_n \|u_{init}^{(n)}\|_{L^2(0, t_{\min}; L^2(\Omega(\ell_{opt}(T_n))))} < \infty. \quad (5.50)$$

Such a control can be determined using the classical method of moments as described for example in [71].

We define the control $\tilde{v}^{(n)}(t) = \begin{cases} u_{init}^{(n)}(t), & t \in (0, t_{\min}), \\ \hat{v}, & t \geq t_{\min}. \end{cases}$

Then $\tilde{v}^{(n)}$ is feasible for $(OCE)(T_n, \Omega(\ell_{opt}(T_n)))$ and thus we have

$$J_{T_n}(\hat{u}_{opt}^{T_n}, \ell_{opt}(T_n)) \leq J_{T_n}(\tilde{v}^{(n)}, \ell_{opt}(T_n)). \quad (5.51)$$

Moreover, (5.48) implies that for all $\varepsilon > 0$ if $T_n > 0$ is sufficiently large we have

$$\lim_{T_n \rightarrow \infty} \frac{1}{T_n} J_{T_n}(\tilde{v}^{(n)}, \ell_{opt}(T_n)) \leq \frac{1}{T_n} J_{T_n}(\hat{u}_{opt}^{T_n}, \ell_{opt}(T_n)) + \varepsilon. \quad (5.52)$$

This can be seen as follows. Since $\lim_{n \rightarrow \infty} T_n = \infty$ due to (5.50) the contribution of the integral on the time interval $(0, t_{\min})$ vanishes in the limit, that is we have

$$\lim_{n \rightarrow \infty} \frac{\int_0^{t_{\min}} \|y(\tilde{v}^{(n)}) - y^d\|_{L^2(\Omega)}^2 + \gamma \|\partial_t(y(\tilde{v}^{(n)}) - y^d)\|_{L^2(\Omega)}^2 + \|\tilde{v}^{(n)} - u^d\|_{L^2(\Gamma)}^2 dt}{T_n} = 0$$

and since $\lim_{T_n \rightarrow \infty} \ell_{opt}(T_n) = \hat{\ell}$ we have

$$\lim_{n \rightarrow \infty} \frac{J_{T_n}(\tilde{v}^{(n)}, \ell_{opt}(T_n))}{T_n} = \lim_{n \rightarrow \infty} \frac{T_n - t_{\min}}{T_n} I(\hat{v}, \ell_{opt}(T_n)) = I(\hat{v}, \hat{\ell}).$$

Assuming the exact controllability of the system in the finite time $t_{\min}$ allows to choose for all $n \in \{1, 2, 3, \dots\}$, a control function $u_{init}^{(n)}(t) \in L^2(0, t_{\min}; L^2(\Omega(\ell_{opt}(T_n))))$ that steers the system to the constant state $y(t_{\min}, \cdot) = \hat{y}$, $y_t(t_{\min}, \cdot) = 0$ and satisfies

$$\max_n \|u_{init}^{(n)}\|_{L^2(0, t_{\min}; L^2(\Omega(\ell_{opt}(T_n))))} < \infty. \quad (5.53)$$

Such a control can be determined using the classical method of moments as described for example in [71].

We define the control $\tilde{v}^{(n)}(t) = \begin{cases} u_{init}^{(n)}(t), & t \in (0, t_{\min}], \\ \hat{v}, & t \geq t_{\min}. \end{cases}$

For all $n \in \{1, 2, 3, \dots\}$ this implies the inequality

$$\frac{1}{T_n} J_{T_n}(\hat{u}_{opt}^{T_n}, \ell_{opt}(T_n)) \leq \frac{1}{T_n} J_{T_n}(\tilde{v}^{(n)}, \ell_{opt}(T_n)).$$

This yields (5.51).

Assume that the optimal shape problem with (OCE) has a solution (ℓ_{opt}, u_{opt}) . Then u_{opt} is a solution of (OCE). Let $\hat{v}_{opt}$ denote the solution of (OCS) for ℓ_{opt} . Then Theorem 10 implies that (5.47) holds.

Proof of Theorem 10. Define $M_F = M_1 + M_2 + M_3$, $M_G = \tilde{M}_1 + \tilde{M}_2$, $M_H = \hat{M}_1 + \hat{M}_2$. Due to (5.23), (5.34), (5.38) and (5.42) for all $k \in \{0, 1, 2, 3, \dots\}$ we have

$$\begin{aligned} |b_{k,T}(t)| &\leq M_F e^{-\frac{\sqrt{\gamma}}{2}(T-t)} |b_{k,T}(T)| \\ &+ [\sqrt{1 + \gamma \lambda_k} M_G |a_k(0)| + M_H |a'_k(0)|] \left[\exp(-\frac{\sqrt{\gamma}}{2} t) + \exp(-\frac{\sqrt{\gamma}}{2}(T-t)) \right]. \end{aligned} \quad (5.54)$$

For the square, this yields the bound

$$\begin{aligned} |b_{k,T}(t)|^2 &\leq 3M_F^2 e^{-\sqrt{\gamma}(T-t)} |b_{k,T}(T)|^2 \\ &+ \left[6(1 + \gamma \lambda_k) M_G^2 |a_k(0)|^2 + 6M_H^2 |a'_k(0)|^2 \right] [\exp(-\sqrt{\gamma}t) + \exp(-\sqrt{\gamma}(T-t))]. \end{aligned}$$

Due to Parseval's equation this implies for all $t \in [0, T]$

$$\begin{aligned} \|\mu^T(t)\|_{L^2(\Omega)}^2 &= \sum_{k=0}^{\infty} |b_k(t)|^2 \\ &\leq \sum_{k=0}^{\infty} 3M_F^2 e^{-\sqrt{\gamma}(T-t)} |b_k(T)|^2 \\ &\quad + [6(1 + \gamma \lambda_k) M_G^2 |a_k(0)|^2 + 6M_H^2 |a'_k(0)|^2] [\exp(-\sqrt{\gamma}t) + \exp(-\sqrt{\gamma}(T-t))]. \end{aligned}$$

Since the initial state (y^0, y^1) satisfies

$$\sum_{k=0}^{\infty} \lambda_k |a_k(0)|^2 + |a'_k(0)|^2 < \infty,$$

this yields an exponential turnpike property for the adjoint state. To be precise, we have for all $t \in [0, T]$

$$\|\mu^T(t)\|_{L^2(\Omega)}^2 \leq \tilde{C}(y_0, y_1, p^\sigma) [\exp(-\sqrt{\gamma}t) + \exp(-\sqrt{\gamma}(T-t))] \quad (5.55)$$

with a real number $\tilde{C}(y_0, y_1, p^\sigma)$ that is independent of T and k .

For the optimal controls, for $\Gamma = \Omega$ we have $\nu^T = \mu^T$, hence we have a similar inequality as (5.55) for $\|\nu^T(t)\|_{L^2(\Omega)}^2$.

Note that in proof of (5.34), (5.38) and (5.42) in the estimates, we did not take advantage of the real part that appears in the representations. In fact, we have always used upper bound for the modulus of the complex number whose real part appears in the expressions. In the representation of the second order derivatives, the only change is that the factor z_k^2 appears in the complex variable representations compared to the original expression. Therefore our proof also yields the inequality

$$|b''_{k,T}(t)|^2 \leq 3|z_k|^2 M_F^2 e^{-\sqrt{\gamma}(T-t)} |b_{k,T}(T)|^2 \quad (5.56)$$

$$+ |z_k|^2 [6(1 + \gamma \lambda_k) M_G^2 |a_k(0)|^2 + 6M_H^2 |a'_k(0)|^2] [\exp(-\sqrt{\gamma}t) + \exp(-\sqrt{\gamma}(T-t))].$$

We have

$$a_k = \frac{-b''_k + (\gamma - \lambda_k) b_k}{1 + \gamma \lambda_k}. \quad (5.57)$$

Define

$$M_\alpha := \sup_k \frac{|z_k|^2}{1 + \gamma \lambda_k} = \sup_k \frac{\sqrt{1 + \lambda_k^2}}{1 + \gamma \lambda_k} < \infty, \quad M_\beta := \sup_k \frac{\lambda_k - \gamma}{1 + \gamma \lambda_k} < \infty$$

and

$$M_\kappa := M_F |b_{k,T}(T)| + \sqrt{1 + \gamma \lambda_k} M_G |a_k(0)| + M_H |a'_k(0)|.$$

Due to (5.57) we also have the following exponential inequality for the coefficients in the expansion of the optimal state:

$$|a_k(t)| \leq (M_\alpha + M_\beta) M_\kappa \left[e^{-\frac{\sqrt{\gamma}}{2}t} + e^{-\frac{\sqrt{\gamma}}{2}(T-t)} \right].$$

This yields the turnpike inequality for the state ω^T :

$$\|\omega^T(t)\|_{L^2(\Omega)}^2 \leq \tilde{C}(y_0, y_1, p^\sigma) [\exp(-\sqrt{\gamma}t) + \exp(-\sqrt{\gamma}(T-t))] \quad (5.58)$$

with a real number $\tilde{C}(y_0, y_1, p^\sigma)$ that is independent of T .

Note that in proof of (5.34), (5.38) and (5.42) in the estimates, we did not take advantage of the structure of the spectrum. In fact, the constants in the turnpike inequalities (5.34) and $M_G = \tilde{M}_1 + \tilde{M}_2$ in (5.38) and $M_H = \hat{M}_1 + \hat{M}_2$ in (5.42) are independent of k .

Remark 11. *From the mathematical point of view, it is of interest to analyze the nucleation of a cycle at the internal node of network. This means that the internal node is replaced by a small cycle, see Figures 5.11, 5.12. The question which internal node is selected can be solved in the steady state case, the topological derivative of the cost is introduced to this end. The domain decomposition method is applied to derive the topological derivative's form. The associated Steklov-Poincaré operator is represented by a matrix $\varepsilon \rightarrow \Lambda(\varepsilon)$ which is semidefinite positive and differentiable at $\varepsilon = 0^+$, the derivative is denoted $\Lambda'(0)$ for the sake of simplicity. In the case of wave equation such a result is not known, however, the numerical experiments show that dependence is regular. Therefore,*

we assume that for a given initial conditions the solution of wave equation enjoys the properties of the steady state boundary value problem for the nucleation of the small cycle. Let us note that for $\varepsilon > 0$ the wave equation is well defined and the regularity of initial conditions required for the turnpike property is already given. Here we assume that the limit of the cost for $\varepsilon \rightarrow 0^+$ is well defined and the value of the cost is continuous $\varepsilon = 0$. This assumption does not imply the differentiability of the cost at $\varepsilon \rightarrow 0^+$.

Assumption 2. Let us consider the wave equation on network and let $\mathcal{J}(\Omega)$ be the cost for given initial conditions y_0, y_1 and given time horizon $T > 0$. At the internal node P_0 of the network a cycle of size $\varepsilon > 0$ is introduced, which leads to the cost $\mathcal{J}(\Omega_\varepsilon)$. The topological variation of the network is admissible provided we have

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{J}(\Omega_\varepsilon) = \mathcal{J}(\Omega).$$

5.5 An Example

Let real numbers $L > 0$, $T > 0$, $c > 0$ and $\gamma > 0$ be given. Let $y_0 \in H^1(0, L)$ with $y_0(0) = 0$ and $y_1 \in L^2(0, L)$, $z \in H^1(0, L)$ with $\zeta = z'(L)$ be given. Consider the problem

$$\min \int_0^T \int_0^L |y(t, x) - z(x)|^2 + \gamma |y_t(t, x)|^2 dx + |u(t) - \zeta|^2 dt$$

subject to

$$\begin{cases} y(0, x) = y_0(x), & x \in (0, L), \\ y_t(0, x) = y_1(x), & x \in (0, L), \\ y_{tt}(t, x) = c^2 y_{xx}(t, x), & (t, x) \in (0, T) \times (0, L), \\ y(t, 0) = 0, & t \in (0, T), \\ y_x(t, L) = u(t), & t \in (0, T). \end{cases} \quad (5.59)$$

The solution to the initial boundary value problem for a control $u \in L^2(0, T)$ is stated in [22], Theorem 2.3, p. 17 in the form

$$y(t, x) = \sum_{n=0}^{\infty} \alpha_n(t) \varphi_n(x) \quad (5.60)$$

with the eigenfunctions $\varphi_n(x) = \frac{\sqrt{2}}{\sqrt{L}} \sin\left(\left(\frac{\pi}{2} + n\pi\right) \frac{x}{L}\right)$, $n \in \{0, 1, 2, \dots\}$. For the eigenvalues we have

$$\lambda_n = \frac{1}{L^2} \left(\frac{\pi}{2} + n\pi\right)^2$$

and the minimal eigenvalue is $\lambda_0 = \frac{\pi^2}{4} \frac{1}{L^2}$. For $n \in \{0, 1, 2, \dots\}$, let

$$\alpha_n^0 = \int_0^L y_0(x) \varphi_n(x) dx, \quad \alpha_n^1 = \int_0^L y_1(x) \varphi_n(x) dx. \quad (5.61)$$

We have

$$\begin{aligned} \alpha_n(t) = & \alpha_n^0 \cos\left(\left(\frac{\pi}{2} + n\pi\right)\frac{t}{t_0}\right) + \alpha_n^1 \frac{t_0}{\frac{\pi}{2} + n\pi} \sin\left(\left(\frac{\pi}{2} + n\pi\right)\frac{t}{t_0}\right) \\ & + (-1)^n c^2 \frac{\sqrt{2}}{\sqrt{L}} \frac{t_0}{\frac{\pi}{2} + n\pi} \int_0^t u(s) \sin\left(\left(\frac{\pi}{2} + n\pi\right)\frac{t-s}{t_0}\right) ds, \end{aligned} \quad (5.62)$$

where $t_0 = \frac{L}{c}$. Since Parseval's identity states that almost everywhere on $[0, T]$, we have

$$\int_0^L y(t, x)^2 dx = \sum_{n=0}^{\infty} |\alpha_n(t)|^2 \quad \text{and} \quad \int_0^L y_t(t, x)^2 dx = \sum_{n=0}^{\infty} |\alpha'_n(t)|^2.$$

We discuss the regularization for the third term of $\alpha_n(t)$ in appendix A.4. We can represent the objective functional in the form

$$\begin{aligned} & \int_0^T \int_0^L |y(t, x) - z(x)|^2 + \gamma |y_t(t, x)|^2 dx + |u(t) - \zeta|^2 dt \\ & = \int_0^T \sum_{n=0}^{\infty} |\alpha_n(t)|^2 + \gamma |\alpha'_n(t)|^2 + |u(t) - \zeta|^2 dt + \int_0^T \int_0^L z^2(x) + \sum_{n=0}^{\infty} 2\alpha_n(t) \varphi_n(x) z(x) dx dt. \end{aligned}$$

If the control space $L^2(0, T)$ is replaced by a finite dimensional space of piecewise constant control functions, this yields a finite dimensional quadratic optimization problem. Since in this case, the necessary optimality conditions are a finite dimensional system of linear equations, this can be used to obtain numerical approximations of the optimal control.

5.5.1 Numerical Solutions

In this section, we discuss the numerical solutions for three examples. We consider the following problem data:

$$c := 1; \quad \gamma := 0.1; \quad T := 1, 10, 100; \quad L := 1;$$

We choose $y_1(x) := 0 \quad (x \in (0, L))$.

1. $y_0(x) := x, z = 0, \zeta = 0$ (example 1);
2. $y_0(x) := \pi^{-1} \sin(\pi x), z = 0, \zeta = 0$ (example 2);
3. $y_0(x) := \pi^{-1} \sin(\pi x), z = x, \zeta = 1$ (example 3);

The coefficients α_n^0 and α_n^1 in (5.61) are obtained as the Fourier coefficients of the chosen functions $y_0(x), y_1(x)$.

In order to solve the optimal boundary control problem numerically, a finite-dimensional approximation is used on two sides of (5.59) simultaneously:

First, the series expansion in the objective functional has to be cut after N terms which leads us to the consideration of the problem $(\mathbf{OPT})(\gamma, N)$. Second, we compute approximations for the optimal controls $u \in L^2(0, T)$ in the space of piecewise constant functions. Let a grid $0 = t_0 < t_1 < t_2 < \dots < t_M = T$ be given. For $i \in \{1, \dots, M\}$ let

$$v_j(t) := \begin{cases} 1 & \text{if } t \in [t_{j-1}, t_j), \\ 0 & \text{elsewhere,} \end{cases}$$

and, define the finite dimensional space $X_M(T)$ by

$$X_M(T) := \text{span}\{v_j(\cdot) : j = 1, \dots, M\}.$$

For any $u \in X_M(T)$ we use the representation

$$u(t) = \sum_{j=1}^M u(t_{j-1})v_j(t), \quad t \in [0, T),$$

where $v_j(t)$ stands for the characteristic function $\chi_{[t_{j-1}, t_j)}(t)$ of the interval $[t_{j-1}, t_j)$ and the approximation of control is defined by the vector $U = \text{col}(U_1, \dots, U_M) \in \mathbb{R}^M$. Hence, we are finally led to solve the problem

$$\begin{aligned} (\mathbf{D}_{\text{opt}})(\gamma, N, M) \min_{u \in X_M(T)} & \sum_{j=1}^M (t_j - t_{j-1}) (u(t_{j-1})^2 + 2u(t_{j-1})\zeta + \zeta^2) \\ & + \sum_{j=1}^M \sum_{n=0}^N \int_{t_{j-1}}^{t_j} |\alpha_n(t)|^2 + \gamma |\alpha'_n(t)|^2 dt \\ & + \sum_{j=1}^M \sum_{n=0}^N \int_{t_{j-1}}^{t_j} \int_0^L 2\alpha_n(t) \varphi_n(x) z(x) + z^2(x) dx dt \end{aligned} \quad (5.63)$$

with $\alpha_j(t)$ as defined in (5.62). Problem

$$(\mathbf{D}_{\text{opt}})(\gamma, N, M)$$

can be equivalently formulated as a quadratic programming problem in $\mathbb{R}^M$

$$\min_{U \in \mathbb{R}^M} U^\top Q U + q^\top U + W$$

where the matrix $Q(N)_{M \times M}$, depending on the fixed number N , and the vector $q \in \mathbb{R}^M$ are to be assembled for fixed N from the cost as stated in (5.63). The assemblage is described below. We use the notation $U_j = u(t_{j-1})$, ($j \in \{1, \dots, M\}$) and take into account a constant term W that is independent of U . We have

$$U^\top Q U + q^\top U + W = \sum_{j=1}^M (t_j - t_{j-1}) (u(t_{j-1})^2 + 2u(t_{j-1})\zeta + \zeta^2)$$

$$\begin{aligned}
& + \sum_{j=1}^M \sum_{n=0}^N \int_{t_{j-1}}^{t_j} |\alpha_n(t)|^2 + \gamma |\alpha'_n(t)|^2 dt \\
& + \sum_{j=1}^M \sum_{n=0}^N \int_{t_{j-1}}^{t_j} \int_0^L 2\alpha_n(t) \varphi_n(x) z(x) + z^2(x) dx dt.
\end{aligned}$$

For the convergence of the approximation, it is important to increase both N and M simultaneously. Otherwise, if only $M \rightarrow \infty$ convergence to the optimal control in general does not occur (due to a possible spillover effect). For $t \in (t_{j-1}, t_j)$ we have

$$\begin{aligned}
\alpha_n(t) &= \alpha_n^0 \cos\left(\left(\frac{\pi}{2} + n\pi\right)t\right) + (-1)^n \frac{\sqrt{2}}{\frac{\pi}{2} + n\pi} \int_0^t u(s) \sin\left(\left(\frac{\pi}{2} + n\pi\right)(t-s)\right) ds \\
&= \alpha_n^0 \cos\left(\left(\frac{\pi}{2} + n\pi\right)t\right) + (-1)^n \frac{\sqrt{2}}{\frac{\pi}{2} + n\pi} \sum_{k=1}^j U_k \int_0^t v_k(s) \sin\left(\left(\frac{\pi}{2} + n\pi\right)(t-s)\right) ds \\
&= \alpha_n^0 \cos\left(\left(\frac{\pi}{2} + n\pi\right)t\right) + (-1)^n \frac{\sqrt{2}}{\frac{\pi}{2} + n\pi} V_j^\top U,
\end{aligned}$$

where $V_j = \text{col}(V_{j,1}, V_{j,2}, \dots, V_{j,j}, 0, \dots, 0)$,

$$V_{j,k} = \begin{cases} \int_{t_{k-1}}^{t_k} \sin\left(\left(\frac{\pi}{2} + n\pi\right)(t-s)\right) ds & \text{if } k < j, \\ \int_{t_{k-1}}^t \sin\left(\left(\frac{\pi}{2} + n\pi\right)(t-s)\right) ds & \text{if } k = j, \\ 0 & \text{if } k > j. \end{cases}$$

and

$$\begin{aligned}
|\alpha_n(t)|^2 &= (\alpha_n^0)^2 \cos^2\left(\left(\frac{\pi}{2} + n\pi\right)t\right) + \frac{2}{\left(\frac{\pi}{2} + n\pi\right)^2} U^\top V_j V_j^\top U \\
&+ (-1)^n \frac{2\sqrt{2}\alpha_n^0}{\frac{\pi}{2} + n\pi} \cos\left(\left(\frac{\pi}{2} + n\pi\right)t\right) V_j^\top U.
\end{aligned} \tag{5.64}$$

This implies

$$\begin{aligned}
\sum_{j=1}^M \sum_{n=0}^N \int_{t_{j-1}}^{t_j} |\alpha_n(t)|^2 &= \sum_{j=1}^M \sum_{n=0}^N (\alpha_n^0)^2 \int_{t_{j-1}}^{t_j} \cos^2\left(\left(\frac{\pi}{2} + n\pi\right)t\right) dt \\
&+ \sum_{j=1}^M \sum_{n=0}^N \frac{2}{\left(\frac{\pi}{2} + n\pi\right)^2} U^\top \int_{t_{j-1}}^{t_j} V_j V_j^\top dt U \\
&+ \sum_{j=1}^M \sum_{n=0}^N (-1)^n \frac{2\sqrt{2}\alpha_n^0}{\frac{\pi}{2} + n\pi} \int_{t_{j-1}}^{t_j} \cos\left(\left(\frac{\pi}{2} + n\pi\right)t\right) V_j^\top dt U \\
&= W_1 + U^\top Q_1 U + q_1^\top U.
\end{aligned} \tag{5.65}$$

Moreover, for the derivatives $\alpha'_n(t)$, we have for $t \in (t_{j-1}, t_j)$

$$\begin{aligned}\alpha'_n(t) &= -\left(\frac{\pi}{2} + n\pi\right)\alpha_n^0 \sin\left(\left(\frac{\pi}{2} + n\pi\right)t\right) + (-1)^n \sqrt{2} \sum_{k=1}^j U_k \int_0^t v_k(s) \cos\left(\left(\frac{\pi}{2} + n\pi\right)(t-s)\right) ds \\ &= -\left(\frac{\pi}{2} + n\pi\right)\alpha_n^0 \sin\left(\left(\frac{\pi}{2} + n\pi\right)t\right) + (-1)^n \sqrt{2} \tilde{V}_j^\top U,\end{aligned}\tag{5.66}$$

where $\tilde{V}_j = \text{col}(\tilde{V}_{j,1}, \tilde{V}_{j,2}, \dots, \tilde{V}_{j,j}, 0, \dots, 0)$,

$$\tilde{V}_{j,k} = \begin{cases} \int_{t_{k-1}}^{t_k} \cos\left(\left(\frac{\pi}{2} + n\pi\right)(t-s)\right) ds & (k < j), \\ \int_{t_{k-1}}^t \cos\left(\left(\frac{\pi}{2} + n\pi\right)(t-s)\right) ds & (k = j), \\ 0 & (k > j). \end{cases}$$

This yields

$$\begin{aligned}|\alpha'_n(t)|^2 &= (\alpha_n^0)^2 \left(\frac{\pi}{2} + n\pi\right)^2 \sin^2\left(\left(\frac{\pi}{2} + n\pi\right)t\right) + 2U^\top \tilde{V}_j \tilde{V}_j^\top U \\ &\quad + (-1)^{n+1} 2\sqrt{2} \alpha_n^0 \left(\frac{\pi}{2} + n\pi\right) \sin\left(\left(\frac{\pi}{2} + n\pi\right)t\right) \tilde{V}_j^\top U.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\gamma \sum_{j=1}^M \sum_{n=0}^N \int_{t_{j-1}}^{t_j} |\alpha'_n(t)|^2 &= \gamma \sum_{j=1}^M \sum_{n=0}^N (\alpha_n^0)^2 \left(\frac{\pi}{2} + n\pi\right)^2 \int_{t_{j-1}}^{t_j} \sin^2\left(\left(\frac{\pi}{2} + n\pi\right)t\right) dt \\ &\quad + \gamma \sum_{j=1}^M \sum_{n=0}^N 2U^\top \int_{t_{j-1}}^{t_j} \tilde{V}_j \tilde{V}_j^\top dt U \\ &\quad + \gamma \sum_{j=1}^M \sum_{n=0}^N (-1)^{n+1} 2\sqrt{2} \alpha_n^0 \left(\frac{\pi}{2} + n\pi\right) \int_{t_{j-1}}^{t_j} \sin\left(\left(\frac{\pi}{2} + n\pi\right)t\right) \tilde{V}_j^\top dt U \\ &= \gamma(W_2 + U^\top Q_2 U + q_2^\top U).\end{aligned}$$

And we have

$$\begin{aligned}\sum_{j=1}^M \sum_{n=0}^N \int_{t_{j-1}}^{t_j} \alpha_n(t) \varphi_n(x) z(x) &= \sum_{j=1}^M \sum_{n=0}^N \int_{t_{j-1}}^{t_j} \alpha_n^0 \cos\left(\left(\frac{\pi}{2} + n\pi\right)t\right) dt \int_0^L z(x) \varphi(x) dx \\ &\quad + \sum_{j=1}^M \sum_{n=0}^N (-1)^n \frac{\sqrt{2}}{\frac{\pi}{2} + n\pi} \int_{t_{j-1}}^{t_j} V_j^\top dt U \\ &= W_3 + q_3^\top U.\end{aligned}\tag{5.67}$$

For the objective function, this implies

$$\begin{aligned} \sum_{j=1}^M (t_j - t_{j-1}) (u(t_{j-1})^2 + 2u(t_{j-1})\zeta + \zeta^2) &:= U^\top Q_3 U + 2q_\tau^\top U + W_\tau, \\ \sum_{j=1}^M \sum_{n=0}^N \int_{t_{j-1}}^{t_j} |\alpha_n(t)|^2 + \gamma |\alpha'_n(t)|^2 dt &:= U + q_1^\top U + W_1 + \gamma(U^\top Q_2 U + q_2^\top U + W_2), \\ \sum_{j=1}^M \sum_{n=0}^N \int_{t_{j-1}}^{t_j} \int_0^L 2\alpha_n(t) \varphi_n(x) z(x) + z^2(x) dx dt &= 2q_3^\top U + W_3. \end{aligned}$$

Thus

$$U^\top Q U + q^\top U + W = U^\top (Q_1 + \gamma Q_2 + Q_3) U + (q_1 + \gamma q_2 + q_3 + q_\tau)^\top U + W_1 + \gamma W_2 + W_3 + W_\tau.$$

Here we use the notation

$$\begin{aligned} Q_1 &= \sum_{j=1}^M \sum_{n=0}^N \frac{2}{(\frac{\pi}{2} + n\pi)^2} \int_{t_{j-1}}^{t_j} V_j V_j^\top dt, \\ Q_2 &= \sum_{j=1}^M \sum_{n=0}^N 2 \int_{t_{j-1}}^{t_j} \tilde{V}_j \tilde{V}_j^\top dt, \\ Q_3 &= \text{diag}(t_1 - t_0, \dots, t_k - t_{k-1}, \dots, t_M - t_{M-1}), \\ q_1 &= \sum_{j=1}^M \sum_{n=0}^N (-1)^n \frac{2\sqrt{2}\alpha_n^0}{\frac{\pi}{2} + n\pi} \int_{t_{j-1}}^{t_j} \cos\left(\left(\frac{\pi}{2} + n\pi\right)t\right) V_j dt, \\ q_2 &= \sum_{j=1}^M \sum_{n=0}^N (-1)^{n+1} 2\sqrt{2}\alpha_n^0 \left(\frac{\pi}{2} + n\pi\right) \int_{t_{j-1}}^{t_j} \sin\left(\left(\frac{\pi}{2} + n\pi\right)t\right) \tilde{V}_j dt, \\ q_3 &= \sum_{j=1}^M \sum_{n=0}^N (-1)^n \frac{\sqrt{2}}{\frac{\pi}{2} + n\pi} \int_{t_{j-1}}^{t_j} V_j^\top dt, \\ q_\tau &= (t_1 - t_0, t_2 - t_1, \dots, t_M - t_{M-1})^\top, \\ W_1 &= \sum_{j=1}^M \sum_{n=0}^N (\alpha_n^0)^2 \int_{t_{j-1}}^{t_j} \cos^2\left(\left(\frac{\pi}{2} + n\pi\right)t\right) dt, \\ W_2 &= \sum_{j=1}^M \sum_{n=0}^N (\alpha_n^0)^2 \int_{t_{j-1}}^{t_j} \sin^2\left(\left(\frac{\pi}{2} + n\pi\right)t\right) dt, \\ W_3 &= \sum_{j=1}^M \sum_{n=0}^N \int_{t_{j-1}}^{t_j} \alpha_n^0 \cos\left(\left(\frac{\pi}{2} + n\pi\right)t\right) dt \int_0^L z(x) \varphi(x) dx. \end{aligned}$$

We employ Matlab for the computational analysis of all examples. Figure 5.1 and Figure 5.2 illustrate the optimal control and state, respectively, for varying values of T in example 1. Notably, as T increases significantly, the control variable u converges to ζ . Moreover, with the increment in T , there is a discernible trend towards stabilization in the norms of both u and y , as evidenced in Figures 5.3. Fig. 5.4-5.6 present the results obtained from example 2, whereas Fig. 5.7 to 5.9 depicts the outcomes of example 3. Additionally, Fig. 5.10 presents the quotient of the optimal value of $(\mathbf{D}_{\text{opt}})(\gamma, N, M)$, which adheres to the turnpike property, thereby providing valuable insights into the behavior of the system concerning varying T values.

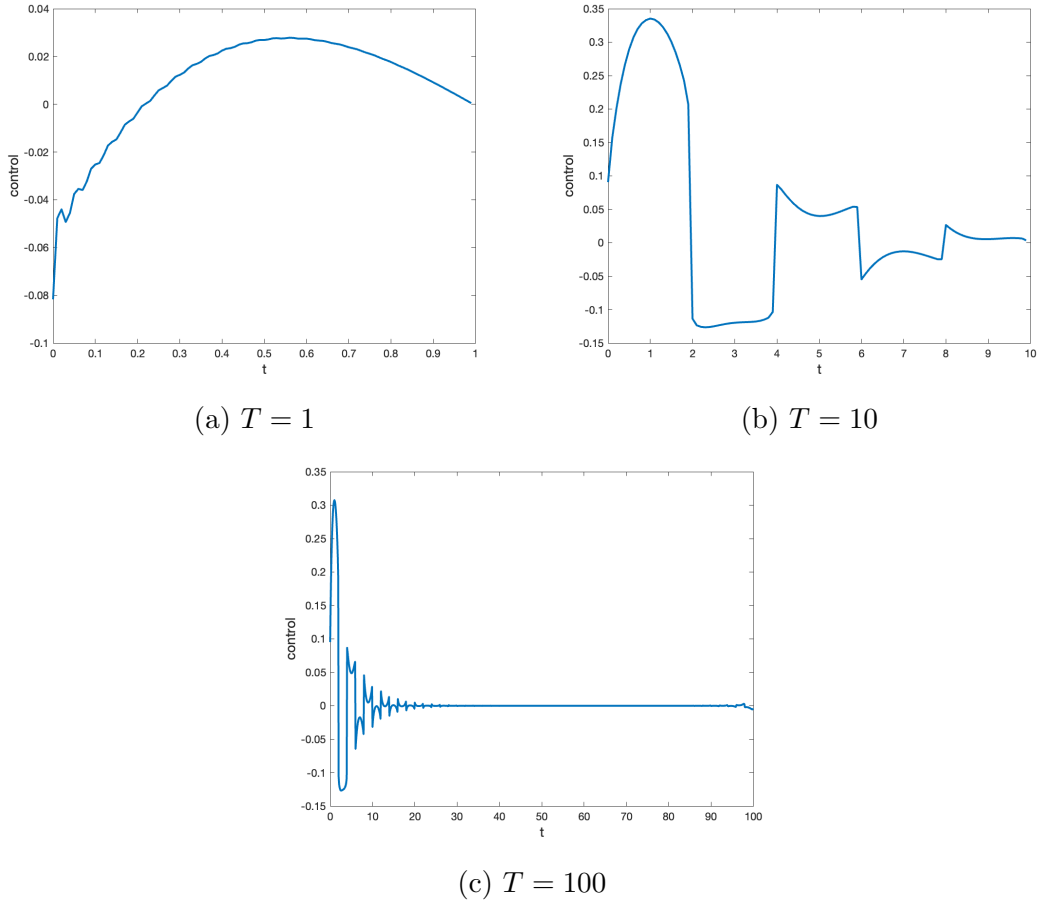
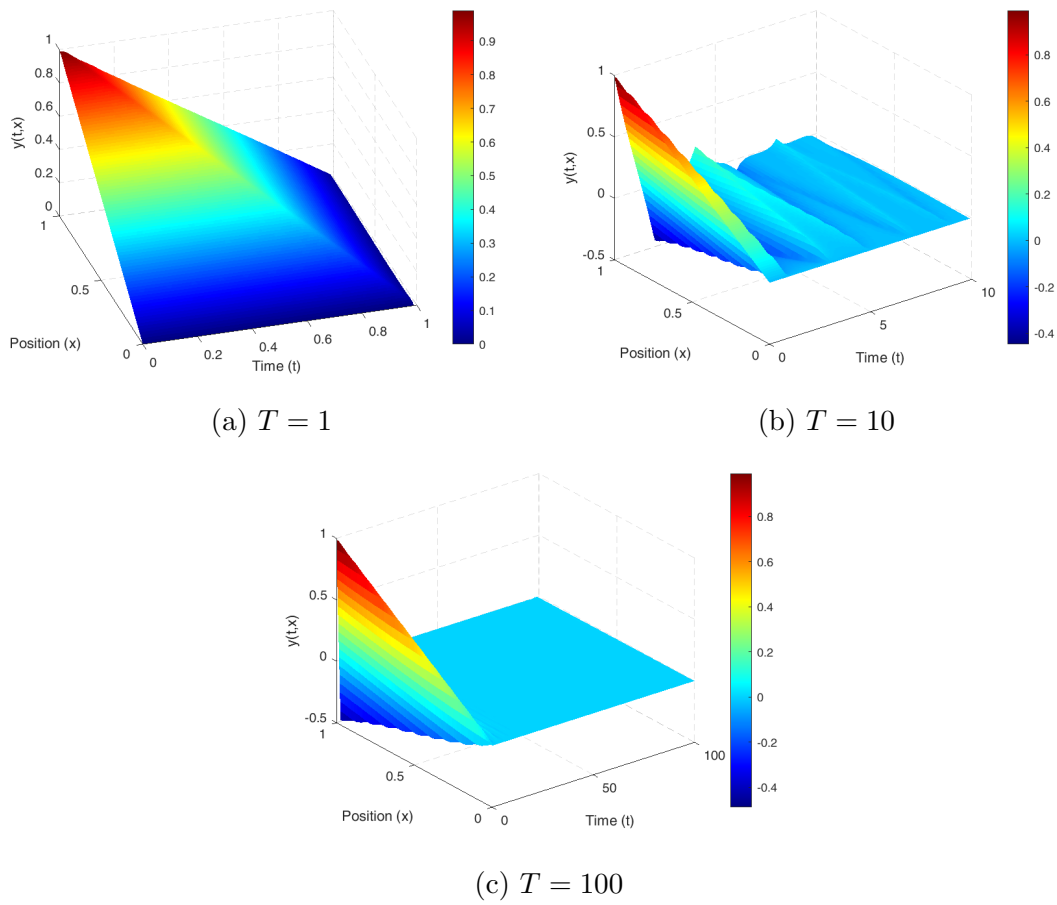
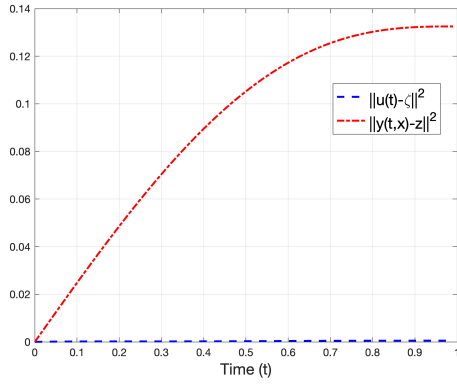
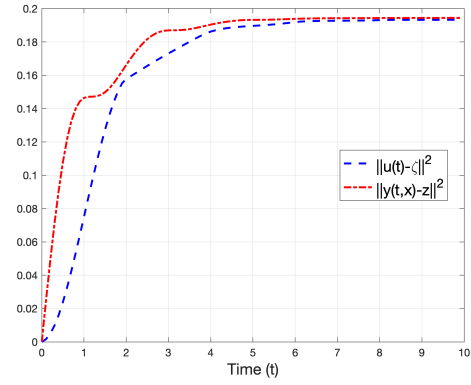
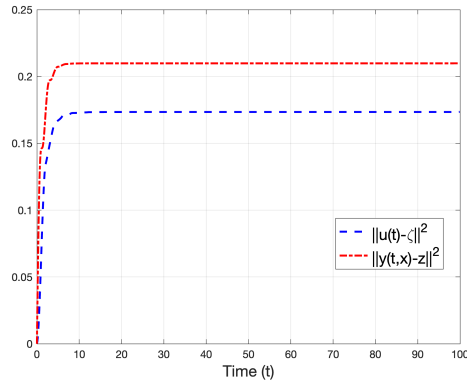
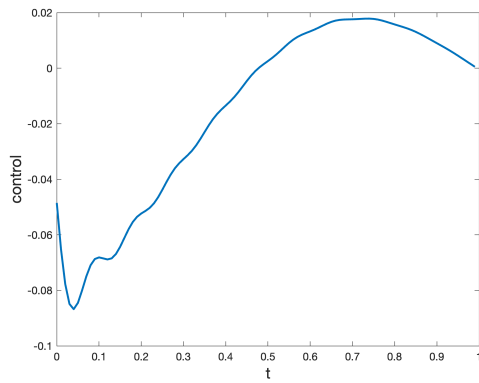
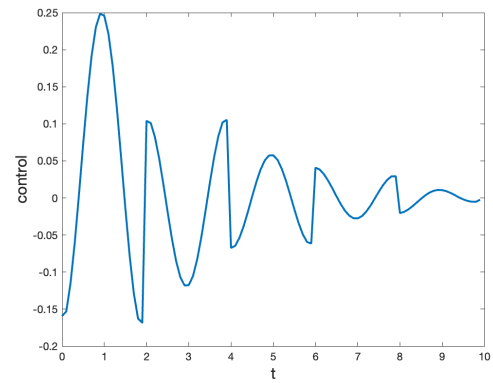
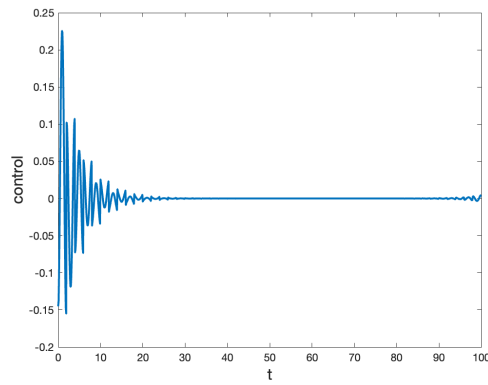
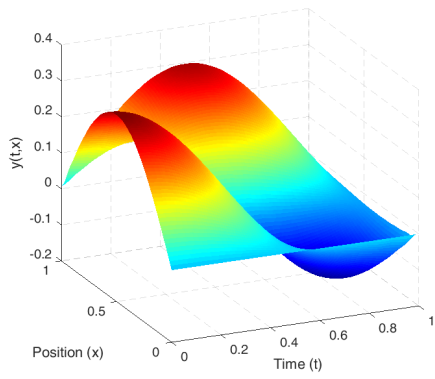
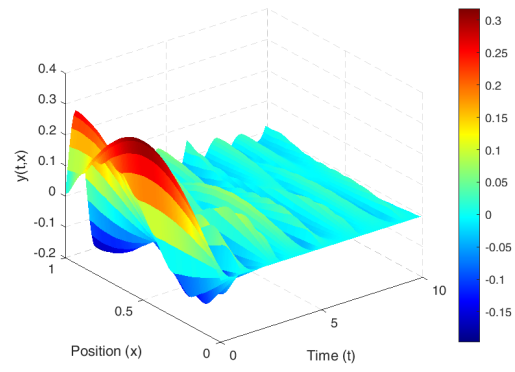
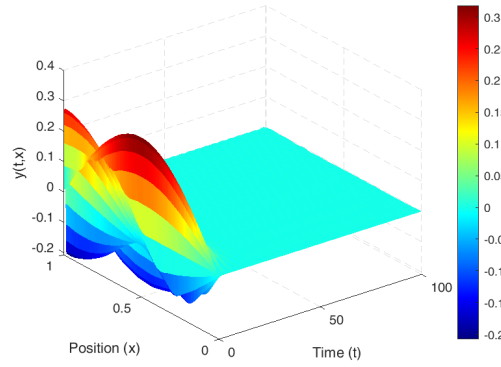


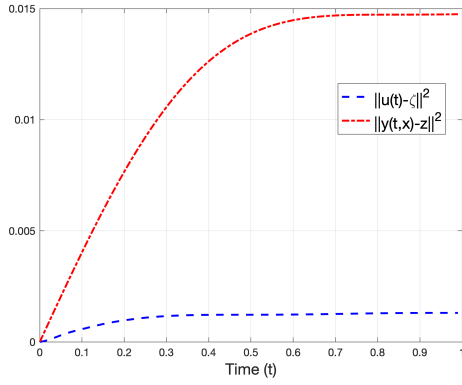
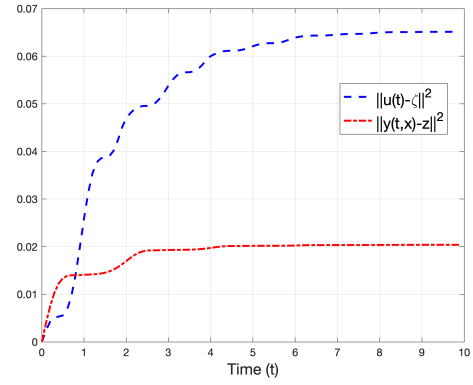
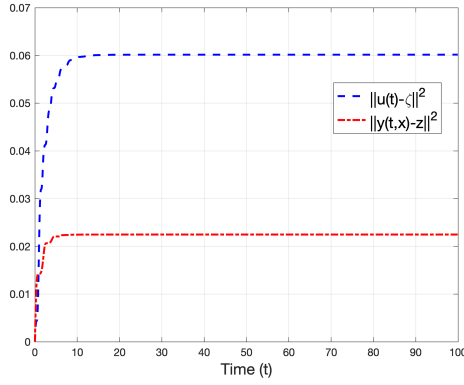
Figure 5.1: Optimal control for different values of T in example 1

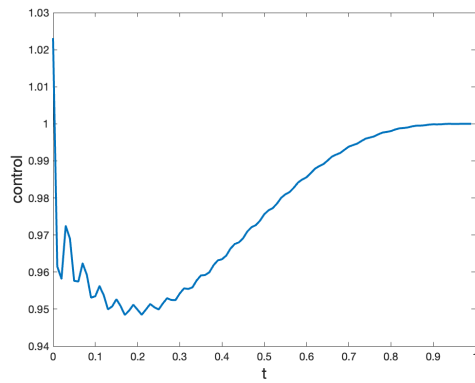
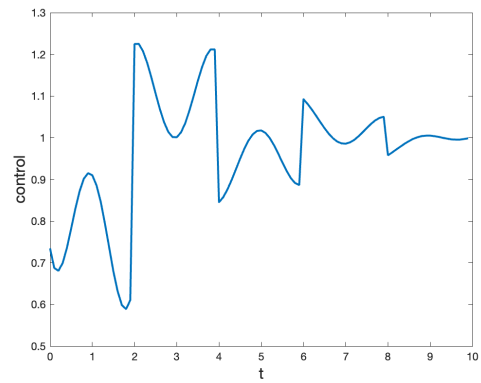
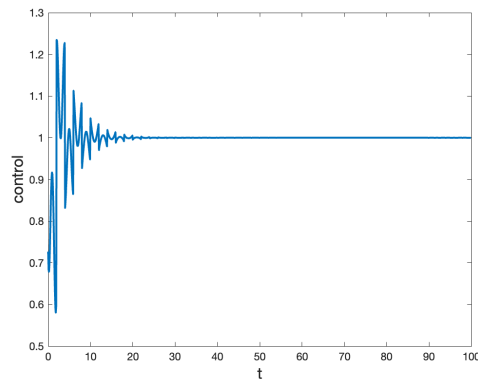
Figure 5.2: Optimal state for different values of T in example 1

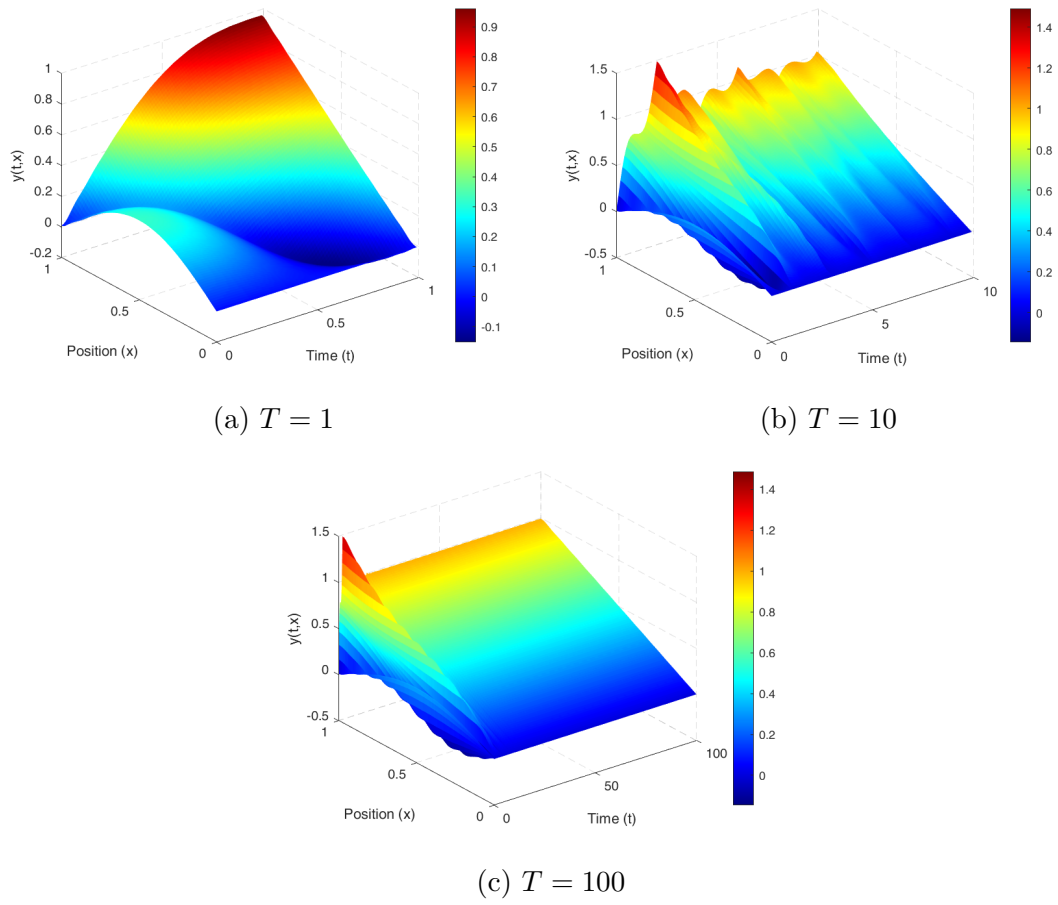
(a) $T = 1$ (b) $T = 10$ (c) $T = 100$ Figure 5.3: Optimal $\|u(t)\|_{L^2}^2$ and $\|y(t,x)\|_{L^2}^2$ for different values of T in example 1

(a) $T = 1$ (b) $T = 10$ (c) $T = 100$ Figure 5.4: Optimal control for different values of T in example 2

(a) $T = 1$ (b) $T = 10$ (c) $T = 100$ Figure 5.5: Optimal state for different values of T in example 2

(a) $T = 1$ (b) $T = 10$ (c) $T = 100$ Figure 5.6: Optimal $\|u(t)\|_{L^2}^2$ and $\|y(t,x)\|_{L^2}^2$ for different values of T in example 2

(a) $T = 1$ (b) $T = 10$ (c) $T = 100$ Figure 5.7: Optimal control for different values of T in example 3

Figure 5.8: Optimal state for different values of T in example 3

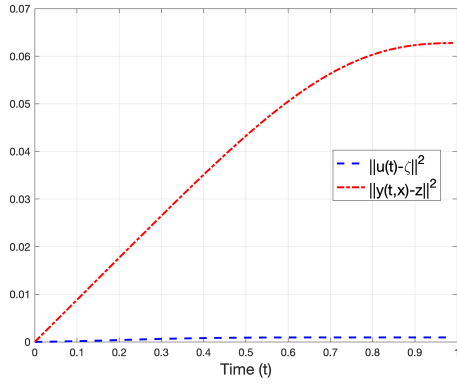
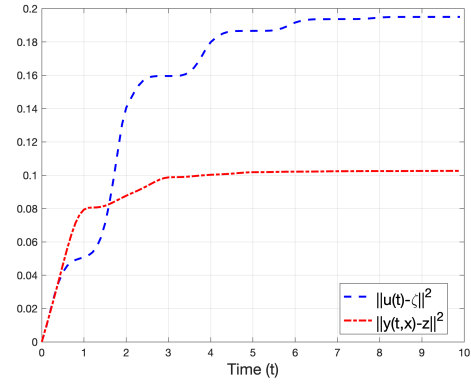
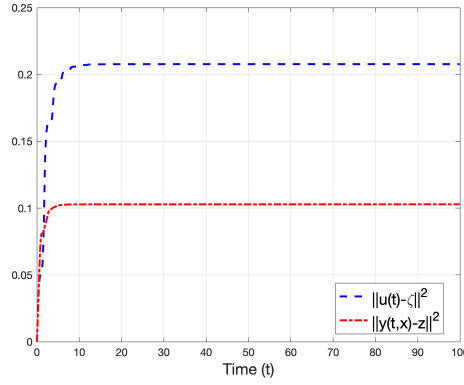
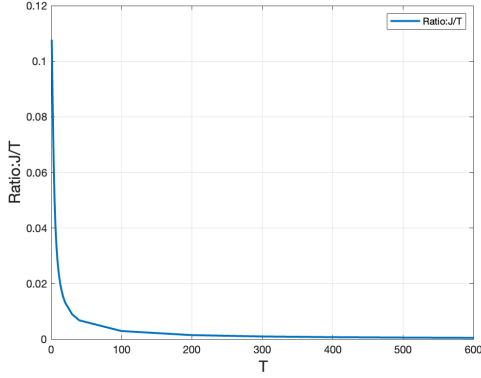
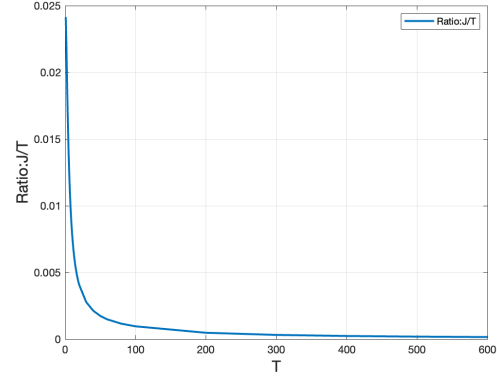
(a) $T = 1$ (b) $T = 10$ (c) $T = 100$

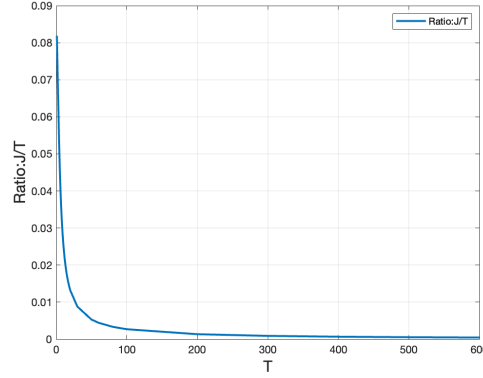
Figure 5.9: Optimal $\|u(t) - \zeta\|_{L^2}^2$ and $\|y(t, x) - z\|_{L^2}^2$ for different values of T in example 3



(a) example 1



(b) example 2



(c) example 3

Figure 5.10: Convergence for different examples

5.6 Topological Derivatives for Network Optimal Control Problems

The topology of network for the purposes of optimal control problems is selected in the framework of the topological derivative method for static problems, we refer to [27] for an elementary example. It turns out, that that the topological derivatives for a class of cost functions can be determined by using the domain decomposition technique for the state equation [79]. We describe in detail the topological derivative method and present numerical results for examples.

Let us consider the network static problem. We define the topological derivatives for optimal cost of network control problems with respect to nucleation of a small cycle. We present also numerical examples. The simplest example of a network is the three-star graph with one central vertex P_0 and three boundary vertices P_1, P_2, P_3 , thus $V = \{P_0, P_1, P_2, P_3\}$ (See Fig. 5.11). There are three edges $E = \{E_1, E_2, E_3\}$.

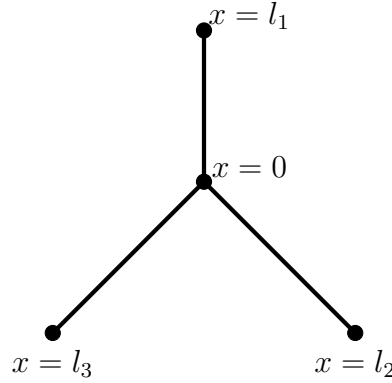
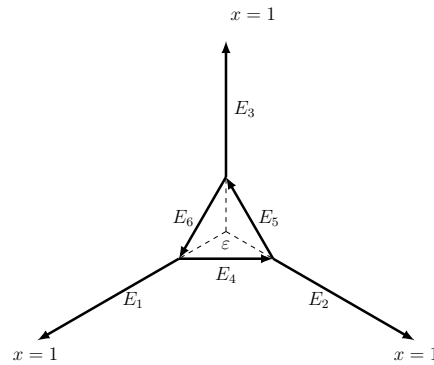


Figure 5.11: The three-star graph

For the steady state problem, singular domain perturbations of the shape are considered. The topological derivatives of the shape functional are defined. The shape and topology optimization is performed. The network is singularly perturbed by a small cycle of the size $\varepsilon \rightarrow 0$ (See Fig. 5.12). In such a case the domain decomposition technique is used and the Steklov-Poincaré operator is introduced. The topological derivative technique is employed in order to decide if a small cycle is useful for the topology optimization of the network.

Figure 5.12: Nucleation of a cycle of size ε in three-star graph

We introduce multiple perturbations of network represented in Figure 5.13. The Steklov-Poincaré operator Λ_ε replaces the subgraph G_ε in the state equation of the network. In this way, the topological derivative of the cost for optimal control problem on the perturbed network is obtained for the nucleation of multiple cycles in the three-star graph.

5.6.1 Shape and Topology Optimization on Networks

We recall briefly the shape and topological derivatives of a given cost for the network. We restrict ourselves to static problems however the shape and topology optimization can

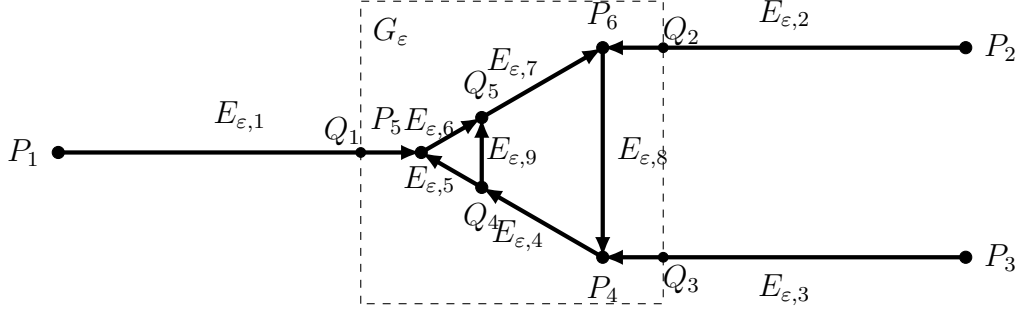


Figure 5.13: Multiple perturbations of the three-star graph for domain decomposition technique.

also be performed for dynamic optimal control problems on networks. For the optimal control problems with the *turnpike Property* the analysis of static problem is useful for the solution of dynamic problem. In particular, the topology of the network is designed using the static problem.

The shape Ω of the network for fixed topology is governed by the finite dimensional vector ℓ , which contains the lengths of edges, $\ell = \text{col}(L_1, \dots, L_N)$ where $N = \#E = \{E_i | i \in \mathcal{I}\}$. Therefore, $\Omega := \Omega(\ell)$, and the cost $\ell \mapsto \mathcal{I}(\ell) := \mathcal{J}(\Omega(\ell))$ is defined by the optimal cost of control problem $\mathcal{J}(\Omega) := J(\hat{u}(\Omega))$ for evolution problem. For the steady state problem the optimal cost of control problem is denoted by $\ell \mapsto J(\hat{v}(\Omega(\ell)))$, where $\hat{v}(\Omega(\ell))$ is steady state optimal control in the domain defined by the shape $\Omega(\ell)$. We consider the Neumann control in the numerical examples presented for the wave equation.

5.6.2 Examples of Topological Derivatives for Networks

Two examples are presented of singular network perturbation by nucleation of a small cycle. In the first example, the topological derivative is evaluated for the optimal control problem with the static state equation, and then for the dynamic state equation, the optimal size of the cycle is determined. In the second example, the topological derivative is evaluated for multiple singular perturbations of the three-star network.

Example for one cycle: In this example the optimal control u is computed for the geometry depicted in Figure 5.14. The variables z and ζ satisfy

$$\begin{cases} -z_i'' = 0, x \in [0, L_i], i = 1, \dots, 6, \\ z_1'(0) = \zeta, z_2(L_2) = z_3(L_3) = 0, \\ \text{Continuity and Kirchhoff Condition.} \end{cases} \quad (5.68)$$

Set $\zeta = 1$, $L_1 = L_2 = L_3 = 2$, $\varepsilon_0 = 0.5$, $\varepsilon_{\max} = 1$, and $0 \leq \varepsilon \leq \varepsilon_{\max}$. Here, $G_{\varepsilon} = \{E_{\varepsilon}, V_{\varepsilon}\}$ contains a small cycle. $E_{\varepsilon} = \{E_{\varepsilon,1}, E_{\varepsilon,2}, \dots, E_{\varepsilon,6}\}$, $V_{\varepsilon} = \{Q_1, Q_2, Q_3, P_4, P_5, P_6\}$, $|E_{\varepsilon,1}| = |E_{\varepsilon,2}| = |E_{\varepsilon,3}| = \varepsilon_{\max} - \varepsilon = 1 - \varepsilon$, $|E_{\varepsilon,4}| = |E_{\varepsilon,5}| = |E_{\varepsilon,6}| = \varepsilon$. The cost functional

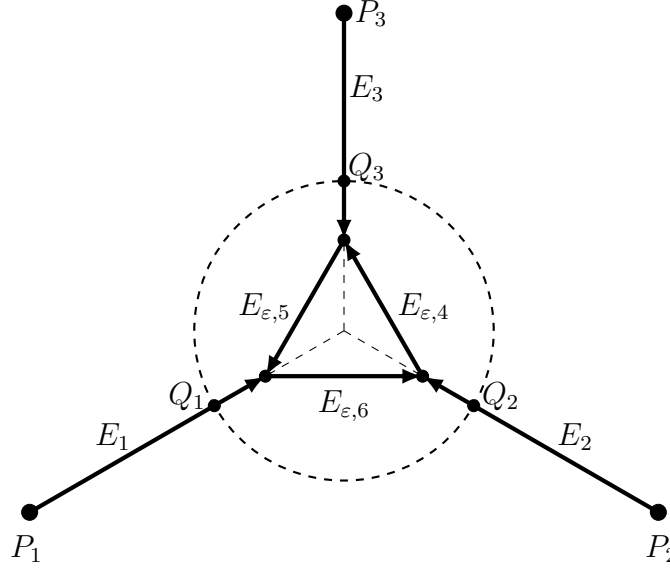


Figure 5.14: Domain decomposition for tripod directed network with an elementary small cycle.

under consideration is defined as:

$$J(u) = \frac{1}{2} \sum_{i=1}^3 \int_0^{L_i - \varepsilon_{\max}} (y_i - z_i)^2 + \frac{1}{2} |u - \zeta|^2.$$

By the Lagrange method, the optimality system is given by

$$\begin{cases} \sum_{i=1}^3 \int_0^{L_i - \varepsilon_{\max}} y_i \phi_i + a(\Omega^0; p, \phi) - \phi(L_i - \varepsilon_{\max})^\top \Lambda_\varepsilon p(L_i - \varepsilon_{\max}) = \sum_{i=1}^3 \int_0^{L_i - \varepsilon_{\max}} z_i \phi_i, \\ a(\Omega^0; y, \phi) - p_1(0) \phi_1(0) - y(L_i - \varepsilon_{\max})^\top \Lambda_\varepsilon \phi(L_i - \varepsilon_{\max}) = -\zeta \phi_1(0), \end{cases}$$

where

$$\phi \in H = \{ \phi_i, \phi'_i \in L^2(0, L_i), \phi_2(0) = \phi_3(0) = 0, \text{ continuity at interior vertices.} \}$$

and

$$\Lambda_\varepsilon = \frac{1}{2\varepsilon - 3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \quad (5.69)$$

Fig. 5.15 shows the shape functional with respect to ε . The derivative of J with respect to ε is consistently negative as ε approaches 0. This is the information that allows for the topology variations by nucleation of a small cycle at an interior vertex of the graph. Another observation emerges: as ε approaches ε_0 , the cost functional (J) converges to 0, indicating an optimal length for the introduced cycle. This implies that at $\varepsilon = \varepsilon_0$,

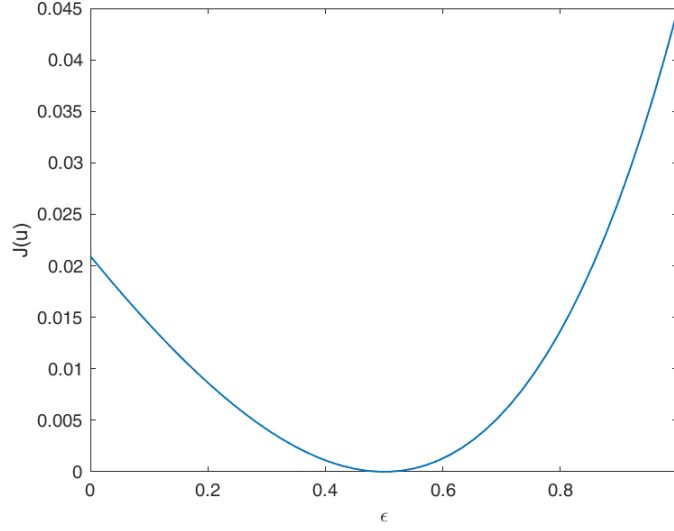


Figure 5.15: The shape functional for $\varepsilon \in [0, 1]$ (Example for one cycle)

the network experiences an optimal configuration, emphasizing the critical nature of this parameter in shaping the network.

Example for multiple cycles: In this example, the geometry is depicted in Figure 5.13. Here, $G_\varepsilon = \{E_\varepsilon, V_\varepsilon\}$ contains a small cycle. $E_\varepsilon = \{E_{\varepsilon,1}, E_{\varepsilon,2}, \dots, E_{\varepsilon,9}\}$, $V_\varepsilon = \{Q_1, Q_2, \dots, Q_5, P_4, P_5, P_6\}$, $|Q_1P_5| = |Q_2P_6| = |Q_3P_4| = \varepsilon_{\max} - \varepsilon = 1 - \varepsilon$, $|E_{\varepsilon,5}| = |E_{\varepsilon,6}| = |E_{\varepsilon,9}| = \varepsilon^2$, $|E_{\varepsilon,4}| = |E_{\varepsilon,7}| = \varepsilon - \varepsilon^2$, $|E_{\varepsilon,8}| = \varepsilon$, and $|P_iQ_i| = L_i - \varepsilon_{\max}$ ($i = 1, 2, 3$). The parameters, cost functional, and optimality system remain consistent with the Example for one cycle (static), with the sole distinction being the form of Λ_ε . The Steklov-Poincaré operator for the small, double cycle is:

$$\Lambda_\varepsilon = -\frac{1}{2\varepsilon - 3} \begin{pmatrix} -2 & 1 & 1 \\ 1 & \frac{-5\varepsilon^2 + 20\varepsilon - 18}{2\varepsilon^2 - 10\varepsilon + 9} & \frac{3\varepsilon^2 - 10\varepsilon + 9}{2\varepsilon^2 - 10\varepsilon + 9} \\ 1 & \frac{3\varepsilon^2 - 10\varepsilon + 9}{2\varepsilon^2 - 10\varepsilon + 9} & \frac{-5\varepsilon^2 + 20\varepsilon - 18}{2\varepsilon^2 - 10\varepsilon + 9} \end{pmatrix}.$$

For numerical results, refer to Fig. 5.16. The topological derivative at $\varepsilon = 0^+$ is negative. And the optimal size of the cycle is $\varepsilon = \varepsilon_0 = 0.5$.

All in all, we exploit the properties of shape and topology optimization problems in one space dimension. The optimal control problems are considered in static and dynamic cases. For the tree network, we show the Turnpike Property for the wave equation and consider the geometric shape optimization. The new case is the tree with a small cycle. In such a case the topology of network is determined by using the topological derivatives obtained in static case. Numerically, the case of the cycle also does not pose a problem. This is new: Also here shape optimization works!

This also holds for a graph with multiple cycles as long as in the shape optimization

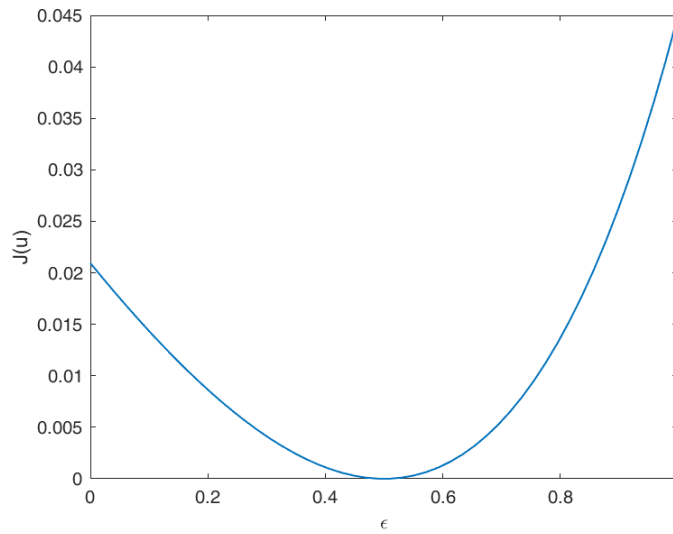


Figure 5.16: The shape functional for the size of cycle $\varepsilon \in [0, 1]$ (Example for multiple cycle).

no topology change occurs, that is no cycle vanishes. The Turnpike property also holds if a cycle disappears. We have shown this under the assumption that the initial state is not supported on the cycle. We expect that this condition is not sharp.

The convergence of gradient flow for shape optimization is also relevant to networks. The modeling of shapes of animals is another possibility for the spectral methods. We refer to [65], and [12] for the related results. Further research will concern the corresponding shape optimization problems in the general three dimensional case.

Chapter 6

Conclusions and Further Research

The complete results, considering Turnpike Property, on control and design of network with scalar wave equation are presented. The topological derivatives are derived for shape functional on networks. The necessary and sufficient optimality conditions are obtained for control and designed networks with nonlinear ODEs. In the linear case, the topological derivatives are used for singular geometric perturbations of network in order to perform the topology optimization. Numerical examples show that this strategy is efficient in improving the cost of optimal control problems.

Further research and open problems:

- Turnpike Property for linear state equations of Timoshenko beams of networks
- Turnpike Property for nonlinear state equations of geometrically exact beams
- Topological derivatives for network of geometrically exact beams in static setting
- Shape and topology optimization of control problems for networks of Timoshenko beams
- Shape optimum design of networks with geometrically exact beams for optimal control problems
- Shape and control of nonlinear state equations in fluid mechanics on networks.

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Acknowledge

Upon finishing this dissertation, I would like to express my great gratitude to all those who have offered me sincere assistance in the past one and a half years.

First and foremost, I sincerely thank the support of the Chinese government and the China Scholarship Council (CSC). Their funding and scholarships enabled me to successfully complete my study abroad program, providing me with invaluable learning and research opportunities. The support from the government and CSC allowed me to receive high-quality education overseas and fully devote myself to academic research. Their support paved the way for me to pursue my academic dreams with greater focus, and I will always remember and strive to repay the care and support from the country and society.

Furthermore, I am deeply grateful to my supervisor Prof. Jan Sokołowski and Prof. Shengfeng Zhu who have given me insightful suggestions and constant encouragement both in my study and in my life. In the many moments of giving up in the process of writing, their unwavering support and guidance have been invaluable, motivating me to persevere and reach new heights in my academic pursuits.

Prof. Sokołowski is an expert in the field of shape topology optimization, and I am deeply honored to be his doctoral student and receive his careful guidance. He is extremely patient. When I encounter confusion, he always patiently answers and even repeatedly explains. His guidance was extremely meticulous, recording all the ideas and deductions in detail in a notebook and handing it to me, demonstrating a rigorous attitude. Although the professor's schedule is very busy, he arranges time every week to guide me. Even if he is not in Poland, we will conduct academic discussions through online platforms.

Prof. Sokołowski not only guides me on weekdays but also values my academic exchange. Whenever he gives academic presentations in other parts of Poland, he invites me to attend together. He also arranged for me to report my research results at a seminar at the research institute and present them at the international IEEE conference. Being invited to give a presentation at the International Conference on Methods and Models in Automation and Robotics (MMAR) was a very meaningful experience for me, as it was my first academic presentation at the conference. In short, thanks to the professor's professional guidance, my research level has been improved and I have had the opportunity to complete this doctoral thesis.

Prof. Sokołowski is very friendly and approachable. He not only took me to visit this city but also shared with me the history of the Old Town. We visited the Royal Castle

together and deeply felt that it had been destroyed and plundered by the armies of Sweden, Brandenburg, Germany, and Russia. We also enjoyed the Bernardo Bellotto 300 Birthday Commemorative Exhibition, commemorating this great artist. And we also attended a concert at the Pałac na Wyspie. On Christmas Eve in 2022, the professor invited me to his flat to celebrate, which allowed me to have a richer experience of European Christmas Eve traditions. On weekdays, the professor often invites me to share coffee and desserts and talk freely about other topics. He is full of love for life and passionate about scenery, music, and history, and his extensive knowledge is admirable.

I would like to express my sincere gratitude to my Chinese supervisor, Professor Zhu Shengfeng, for his unwavering support and encouragement, which enabled me to realize my wish to study abroad. His efforts in connecting me with the eminent scholar Jan Sokolowski in the field of shape optimization provided me with the invaluable opportunity to receive guidance and elevate my academic prowess. The mentorship and care provided by Prof. Zhu are treasures on my academic journey, and I will cherish this valuable experience.

Upon arriving in Poland, Prof. Zhu not only provided academic guidance but also extended caring support in daily life. His invaluable experience and unwavering encouragement helped me navigate challenges and smoothly adapt to the new culture and lifestyle. As both a scholarly guide and personal role model, Prof. Zhu consistently engaged in insightful discussions, fostering my intellectual growth and problem-solving skills. His trust and encouragement propelled me to pursue academic excellence, while his generous support, including financial assistance for my doctoral studies in Poland, alleviated any concerns.

I'm profoundly grateful for Prof. Zhu's selfless dedication and steadfast belief in my potential. His mentorship continues to inspire me, and I'm committed to honoring his guidance by striving for continued success. Additionally, the memorable time when Prof. Zhu visited Warsaw, Poland from January 15th, 2024, to February 5th, 2024, further strengthened our bond and provided invaluable insights and inspiration for my research work. His visit was marked by deep academic exchanges and discussions, during which he offered invaluable advice that has shaped the direction of my future research endeavors. This cherished experience has not only enriched my knowledge but also served as a significant milestone in my academic journey.

Thanks to Prof. Jan W. Owsinski, who is the deputy director of the institute. Before I came, he helped me a lot, including delivering kinds of documents, renting the flat, picking me up at the airport and so on. He also cares about me very much in the institute.

Thanks to my colleagues in the IBS PAN. Thanks to Giovanni Bruccola and The Hung Tran. We visited Centrum Nauki Kopernik in the spring, went to the campfire party in the summer, watched Disney's 100th Anniversary show in the Autumn, and had a hotpot in the winter. Especially during the summer, we had a BBQ at Giovanni's flat. It was a good memory. Wishing that we all successfully obtain our degrees and become true mathematicians.

Thanks to Katarzyna Szulc, who was very concerned about my life when I just arrived

in Warsaw. She is a kind lady. Thanks to Krzysztof Rutkowski, who organized the Chinese New Year for us. He is a nice man. Thanks to Bartłomiej Solarz-Niesłuchowski, who helped me a lot when my laptop was broken and helped me apply for the resident card. He is a humorous man.

Thanks to Qiang Huo, who is a joint Ph.D. student at the Mathematic Institute (IM PAN). We have known each other while preparing for visas. We exchanged information and prepared to go abroad. Our first meeting was near the Christmas tree in the old town of Warsaw. Sometimes we strolled together, cooked dishes, and witnessed his culinary skills improving. We were in many restaurants and explored food here. We celebrated together during Chinese New Year, Lantern Festival, Mid-Autumn Festival, and birthdays, each occasion filled with warmth and joy. We also frequently studied together, and his passion for learning deeply inspired me. The happy moments in Warsaw always seemed fleeting. Qiang is soon returning to Beijing. I wish him to graduate as expected. I hope that we will cross paths again.

Thanks to Mengyu Li, who is also a joint Ph.D. student but in Roma. It was a good memory to travel with you to Rome, Pisa, Florence, Venice, Warsaw, Prague, and Vienna. We have seen many beautiful scenery and museums and took good photos. During my first year in Poland, she always chatted with me, especially at night. The nights in Poland are so long, but fortunately with your company.

Thanks to Yaqin Lyu, a friend that I knew in Poland. Her kindness shone through when she prepared hot ginger cola and took a one-hour bus to deliver it to me when I had a high temperature. I experienced a profound sense of depression and constant fear and loneliness after that illness, and your visits were a source of great comfort. I'm glad you prepared a graduation gift for me. I am sincerely thankful for your friendship and encouragement.

Thanks to Gege Pan, one of my best friends in Warsaw. On the first night that I arrived in Warsaw, she sought me out at my place of residence, accompanied me to the mall to assist with getting a SIM card, and more. We play badminton, go to the gym, take walks in the park, and make dumplings together. These experiences have fostered a deep friendship between us. I am deeply touched by her kindness and assistance.

Thanks to Maomao Ling, one of my friends in Warsaw. We met at the badminton court. He patiently taught me how to play, which helped me improve. Although we hadn't known each other for long, we exercised and went to a shooting club. He is good at time management and improves efficiency. His attitude towards life and work motivates me and gives me a good mindset.

Thanks to Yixin Tan, my friend in China, her arrival in Poland has added vitality to my life. We share the same office, motivating each other to study and creating a conducive learning atmosphere. Her earnest dedication to learning and her diligent spirit of inquiry deeply influence me, inspiring me to strive harder in my studies and personal growth. I wish her all the best in her life and studies in Warsaw. I believe that someone to whom her are important will bring her closer to her aim.

Thanks to my Polish friends Łukas and Kaja, who are not only my former landlords

but also my friends. Upon my arrival in Poland, during those initial days, their invaluable assistance proved instrumental in acquainting me with the local surroundings and imparting culinary skills. Thanks for their help, which enabled me to survive in a foreign country. I will remember the time when we skated and tasted wine together. Our excursion to Miedzyzdroje in August, a quaint town by the Baltic Sea, stands out as a highlight, as we explored its charm and crossed the German border to stroll through a neighboring town. The end of the year saw us celebrating Christmas Day and ushering in the New Year together, marked by joyous moments of fireworks and relishing traditional Polish cuisine. These cherished times will forever hold a special place in my heart.

Last my thanks would go to my beloved parents for this loving consideration and great confidence in me. They support me and love me even when I least deserve to be loved. In addition, many thanks to my friends who have kindly provided unwavering support and companionship during the past two years.

Appendix A

Program

The topological derivative method can be used for the numerical solutions of optimum design problems for networks. The examples with Matlab codes show the constructive way to proceed.

A.1 Nonlinear Beams

This section provides the MATLAB implementation for the nonlinear beam example discussed in Example 2 in section 3.3.2. The objective is to optimize control problems for a nonlinear beam system, using the Newton method to handle the nonlinear terms efficiently and achieve optimal control.

```
1 %% This is a construct example for nonlinear beam
2 % +-----+
3 % | phi(r) = -E(x)r + L(r)Cr; |
4 % | z1.hat = -x+1; |
5 % | zi.hat = 0; i=2,3,4,5,6 |
6 % | f1.hat = -1; |
7 % | fi.hat = 0; i=2,3,4,5,6 |
8 % | |
9 % | z'(x) = phi(z) + e1(f.hat - p), |
10 % | -p'(x) = p(x)phi'(z(x)) + z(x) -z.hat, |
11 % | z(1) = 0, p(0) = 0; |
12 % | P2 element |
13 % +-----+
14
15 function J = optControl(L)
16 if nargin == 0
17     L = 1; % L \in [1/2,3/2]
18 end
19
20 % Define optimization parameters
21 Lopt = 1; % Optimal beam length
22 Lmax = 3/2; % Maximum allowable beam length
```

```

23 hmax = 0.01;
24 N01 = ceil((L)/hmax);
25 Nx = 2*N01 + 1;
26
27 % Calculate number of nodes for optimal and maximum beam lengths
28 Nopt = Lopt/hmax;
29 Nmax = Lmax/hmax;
30
31 % Generate node coordinates for different beam lengths
32 x01 = linspace(0,L,N01+1);
33 xopt = linspace(0,Lopt,Nopt+1)';
34 xmax = linspace(0,Lmax,Nmax+1)';
35
36 % Compute stiffness matrices and mass matrix for finite element analysis
37 [Pb,A1,A2] = graph1D.matirx_P2(N01,x01);
38 [Pbmax,A1max,~] = graph1D.matirx_P2(Nmax,xmax);
39
40 [Ks1,Ks2] = deal(cell(6,6));
41
42 for i = 1 : 6
43     for j = 1 : 6
44         Ks1{i,j} = zeros(size(A1));
45         Ks2{i,j} = zeros(size(A1));
46     end
47 end
48
49 for i = 1 : 6
50     Ks1{i,i} = A2;
51     Ks2{i,i} = A2;
52 end
53
54 Ks1{5,3} = A1;
55 Ks1{6,2} = -A1;
56
57 Ks2{5,3} = -A1;
58 Ks2{6,2} = A1;
59
60 for i = 1 : 6
61     Ks1{i,i}(1,1) = Ks1{i,i}(1,1) + 1;
62     Ks2{i,i}(end,end) = Ks2{i,i}(end,end) - 1;
63 end
64
65 M = deal(cell(6,6));
66 for i = 1 : 6
67     for j = 1 : 6
68         M{i,j} = zeros(size(A1));
69     end
70 end
71
72 for i = 1 : 6
73     M{i,i} = A1;

```

```

74 end
75
76 % Convert cell arrays to numerical matrices
77 Ks1_matrix = cell2mat(Ks1);
78 Ks2_matrix = cell2mat(Ks2);
79 M_matrix = cell2mat(M);
80
81 A = [-Ks1_matrix M_matrix;-M_matrix Ks2_matrix];
82
83 %% Dirichlet condition
84 bdyL = (1:12) * Nx;
85 bdy0 = 1 + (0:11)*Nx;
86
87 %% control
88 xopt = Pbmax';
89
90 fhat = -ones(size(xopt))*1; % Desired force distribution
91 zhat = -xopt+1;
92
93 % Calculate force and state distributions
94 Fhat = Almax*fhat;
95 Zhat = Almax*zhat;
96
97 Fhat(Nx+1:end) = [];
98 Zhat(Nx+1:end) = [];
99
100 % Construct control force vector F
101 F = zeros(12*Nx,1);
102 F(bdy0(1):bdyL(1)) = Fhat;
103 F(bdy0(7):bdyL(7)) = -Zhat;
104
105 C = diag([10^4,10^4,10^4,500,500,500])^(-1);
106
107 %% Initialization of the state
108 % Consistent structural linearisation in flexible-body dynamics with ...
109 %   large rigid-body motion
110 % Henrik Hesse, Rafael Palacios, 2012
111 EA = 10^4; GAs = 10^4;
112 EI = 500; GJ = 500;
113 rhoA = 1; rhoJ = diag([20, 10, 10]);
114 massMat = blkdiag(rhoA*eye(3), rhoJ); % the MASS matrix
115 flexMat = inv(diag([EA, GAs, GAs, GJ, EI, EI])); % the FLEXIBILITY matrix
116
117 Gamma0 = [0; 0; 0]; % there is no initial shear
118 W0hat = [0, -1/sqrt(2), 0; 1/sqrt(2), 0, 1/sqrt(2); 0, -1/sqrt(2), 0];
119 Upsilon0 = func.vec(W0hat);
120
121 z0 = [Gamma0; Upsilon0]; % strains
122 z0 = flexMat\z0; % corresponding stresses
123
124 Y0 = zeros(6*Nx, 1);

```

```

124 p0 = zeros(6*Nx,1);
125
126 wm = [Y0;p0]; % initialize wm
127
128 disp('Solving the semilinear system..');
129 %%% M.A. code %%%
130 tolzero=1e-12;
131 reltolX=1e-6;
132 tolF=1e-15;
133 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
134 iter = 0;
135
136 while 1
137     %%% Newton method: the scheme reads %%%
138     % wm1 = wm - (JacFk(wm))^{-1} Fk(wm) %
139     %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
140     MLmartix = MCL(wm,Nx,C,L,N01);
141     MGmartix = MG(wm,Nx,C,L,N01);
142     NL = blkdiag(MLmartix,MGmartix);
143     Fk_wm = A*wm - NL*wm - F;
144     % boundary condition
145     Fk_wm(bdyL(1:6),1) = 0;
146     Fk_wm(bdy0(7:12),1) = 0;
147
148     JacFk_wm = A - NL;
149     for i = 1 : 6
150         JacFk_wm(bdyL(i), :) = 0;
151         JacFk_wm(bdyL(i), bdyL(i)) = 1;
152         JacFk_wm(bdy0(i+6), :) = 0;
153         JacFk_wm(bdy0(i+6), bdy0(i+6)) = 1;
154     end
155
156     wm1 = wm - JacFk_wm\Fk_wm;
157
158     %%% M.A. code %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
159     rel_err = (wm1 - wm)./wm; %
160     nan_or_inf = find( isnan(rel_err) + isinf(rel_err) ... %
161         + (abs(wm)<=tolzero) ); %
162     rel_err(nan_or_inf) = 0; %
163     if (norm(rel_err,inf) <= reltolX) && (norm(Fk_wm,inf) <= tolF)%
164         break %
165     end %
166     %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
167
168     wm = wm1;
169     iter = iter + 1;
170 end
171
172 z_exp = wm1;
173 z_solu = cell(6,1);
174 for i = 1 : 6

```

```

175     z_solu{i,1} = zeros(Nx,1);
176     z_solu{i,1} = z_exp(bdy0(i):bdyL(i));
177 end
178
179 p_solu = cell(6,1);
180 for i = 7 : 12
181     p_solu{i-6,1} = zeros(Nx,1);
182     p_solu{i-6,1} = z_exp(bdy0(i):bdyL(i));
183 end
184
185 f_solu = cell(6,1);
186 for i = 1 : 6
187     f_solu{i,1} = zeros(Nx,1);
188     f_solu{1,1} = fhat(1:Nx)-p_solu{1,1};
189 end
190
191 f_err = trapz(Pb, (f_solu{1,1}-fhat(1:Nx)).^2);
192 z_err = trapz(Pb, (z_solu{1,1}-zhat(1:Nx)).^2);
193
194 J = f_err + z_err + (L-Lopt)^2 ;
195 J = J/2;
196 disp(['f_err = ', num2str(f_err)]);
197 disp(['z_err = ', num2str(z_err)]);
198 disp(['J = ', num2str(J)]);
199
200 end

```

A.2 Timoshenko Beam

In this section, we present the exact solution and the optimal control code for both the 1D and 3D Timoshenko models.

The first file contains the MATLAB code implementing the exact solution for the 1D Timoshenko model in section 4.6.1. This solution enables precise calculation of displacements and stresses within the structure under specified loading and boundary conditions. The OptControl function presents the optimal control problem code shown for the 1D/3D Timoshenko model in section 4.10.

```

1  % This is to get the exact solution of the 1D model
2  clear
3
4  % parameters
5  % elastic modulus
6  % em = 2.1e11;
7  % % shear modulus
8  % sm = 8.1*1e10 ;
9  % % cross-section area
10 % cs = 0.01 ;

```

```

11 % % torsion constant
12 % tor = 1.41*1e-5 ;
13 % % second moments
14 % secm = 8.33*1e-6 ;
15 % % axial stiffness
16 % kx = cs*em ;
17 % % shear stiffness
18 % kappa = 0.850 ;
19 % ks = kappa*sm*cs ;
20 % % torsional stiffness
21 % cx = sm*tor ;
22 % % bending stiffness
23 % cy = secm*em ;
24 % cz = secm*em ;
25
26 ks = 688500000;
27 cy = 1749300;
28
29 epsVal = 0.5;
30
31 Len = 2;
32
33 L = [Len-epsVal,Len-epsVal,Len-epsVal,epsVal,epsVal,epsVal];
34
35 syms x;
36 f = piecewise(x <= Len-1.00000, 2e6, x > Len-1.00000 & x <= L(2), 0);
37 fz = [0,-f,0,0,0,0];
38 my = [0,0,0,0,0,0];
39 rz = [0,0,0,0,0,0];
40 phiy = [0,0,0,0,0,0];
41 for i = 1 : 3
42     q{i} = [fz(i);my(i)];
43     u0{i} = [rz(i);phiy(i)];
44 end
45
46 for i = 1 : 6
47     D1E{i} = D1(ks,L(i),cy);
48     D2E{i} = D2(ks,L(i),cy);
49     D3E{i} = D3(ks,L(i),cy);
50     D4E{i} = D4(ks,L(i),cy);
51     dE{i} = d(fz(i),my(i),cy,L(i));
52     wE{i} = w(fz(i),my(i),ks,cy,L(i));
53     RE{i} = R(cy,ks,L(i));
54     QE{i} = Q(cy,ks,L(i));
55 end
56
57 A = [D4E{1}+D4E{4}-D1E{5}, -D2E{5}, D3E{4};
58     D3E{5}, D4E{2}+D4E{5}-D1E{6}, -D2E{6};
59     -D2E{4}, D3E{6}, D4E{3}+D4E{6}-D1E{4}]; %symmertic
60
61 b = sym(zeros(6,1));

```

```

62 b(1:2,1) = - q{1} - D3E{1}*u0{1} ...
63           - D4E{1}*subs(wE{1},L(1)) - D4E{4}*subs(wE{4},L(4)) + ...
           D2E{5}*subs(wE{5},L(5)) ...
64           + subs(dE{1},L(1)) + subs(dE{4},L(4));
65 b(3:4,1) = - q{2} - D3E{2}*u0{2} ...
66           - D4E{2}*subs(wE{2},L(2)) - D4E{5}*subs(wE{5},L(5)) + ...
           D2E{6}*subs(wE{6},L(6)) ...
67           + subs(dE{2},L(2)) + subs(dE{5},L(5));
68 b(5:6,1) = - q{3} - D3E{3}*u0{3} ...
69           - D4E{3}*subs(wE{3},L(3)) - D4E{6}*subs(wE{6},L(6)) + ...
           D2E{4}*subs(wE{4},L(4)) ...
70           + subs(dE{3},L(3)) + subs(dE{6},L(6));
71
72 solu = A\b;
73 uL{1} = subs(solu(1:2),L(1));
74 uL{2} = subs(solu(3:4),L(2));
75 uL{3} = subs(solu(5:6),L(3));
76
77 u0{4} = uL{3};
78 u0{5} = uL{1};
79 u0{6} = uL{2};
80
81 uL{4} = uL{1};
82 uL{5} = uL{2};
83 uL{6} = uL{3};
84
85 for i = 1 : 6
86     u{i} = QE{i}*u0{i} + RE{i}*(subs(uL{i},L(i))+subs(wE{i},L(i)))-wE{i};
87 end
88
89 save('u_data.mat', 'u','Len');
90 save('fz.mat', 'fz');
91 save('my.mat', 'my');
92 close all
93
94 figure
95 subplot(2,1,1)
96 fplot(u{1}(1),[0 L(1)])
97 hold on
98 fplot(u{2}(1),[0 L(2)])
99 hold on
100 fplot(u{3}(1),[0 L(3)])
101 legend('u_1','u_2','u_3')
102 title('r.z')
103
104 % figure
105 subplot(2,1,2)
106 fplot(u{1}(2),[0 L(1)])
107 hold on
108 fplot(u{2}(2),[0 L(2)])
109 hold on

```

```

110 fplot(u{3}(2), [0 L(3)])
111 legend('u_1', 'u_2', 'u_3')
112 title('phi_y')
113
114 figure
115 subplot(2,1,1)
116 fplot(u{4}(1), [0 L(4)])
117 hold on
118 fplot(u{5}(1), [0 L(5)])
119 hold on
120 fplot(u{6}(1), [0 L(6)])
121 legend('u_4', 'u_5', 'u_6')
122 title('r_z')
123
124 % figure
125 subplot(2,1,2)
126 fplot(u{4}(2), [0 L(4)])
127 hold on
128 fplot(u{5}(2), [0 L(5)])
129 hold on
130 fplot(u{6}(2), [0 L(6)])
131 legend('u_4', 'u_5', 'u_6')
132 title('phi_y')
133
134 function muy = mu(ks,L,cy)
135     muy = ks/(ks*L^3+12*cy*L);
136 % muz = ks/(ks*L^3+12*czy*L);
137 end
138
139 function Eil = E(t)
140     Eil = [cos(t) sin(t);
141           -sin(t) cos(t)];
142 end
143
144 function D1i = D1(ks,L,cy)
145     muy = mu(ks,L,cy);
146     D1i = zeros(2,2);
147     D1i(1,1) = -12*cy*muy;
148     D1i(2,2) = -cy*(1/L+3*muy*L^2);
149     D1i(2,1) = 6*cy*muy*L;
150     D1i(1,2) = 6*cy*muy*L;
151 end
152
153 function D2i = D2(ks,L,cy)
154     muy = mu(ks,L,cy);
155     D2i = zeros(2,2);
156     D2i(1,1) = 12*cy*muy;
157     D2i(2,2) = cy*(1/L-3*muy*L^2);
158     D2i(2,1) = -6*cy*muy*L;
159     D2i(1,2) = 6*cy*muy*L;
160 end

```

```

161
162 function D3i = D3(ks,L,cy)
163     muy = mu(ks,L,cy);
164     D3i = zeros(2,2);
165     D3i(1,1) = -12*cy*muy;
166     D3i(2,2) = -cy*(1/L-3*muy*L^2);
167     D3i(2,1) = -6*cy*muy*L;
168     D3i(1,2) = 6*cy*muy*L;
169 end
170
171 function D4i = D4(ks,L,cy)
172     muy = mu(ks,L,cy);
173     D4i = zeros(2,2);
174     D4i(1,1) = 12*cy*muy;
175     D4i(2,2) = cy*(1/L+3*muy*L^2);
176     D4i(2,1) = 6*cy*muy*L;
177     D4i(1,2) = 6*cy*muy*L;
178 end
179
180 function [fInt,fInt2,fInt3] = Int(f,L)
181     syms x;
182     fInt = int(f,x,0,x);
183     fInt2 = int(fInt,x,0,x);
184     fInt3 = int(fInt2,x,0,x);
185 end
186
187 function gyi = gy(fz,my,cy,L)
188     [fzInt1,~,~] = Int(fz,L);
189     gyi = 1/cy*(my+fzInt1);
190 end
191
192 function wi = w(fz,my,ks,cy,L)
193     wi = sym(zeros(2,1));
194     gyi = gy(fz,my,cy,L);
195     [~,fzInt2,~] = Int(fz,L);
196     [~,gyInt2,gyInt3] = Int(gyi,L);
197     wi(1) = 1/ks*fzInt2 - gyInt3;
198     wi(2) = gyInt2;
199 end
200
201 function di = d(fz,my,cy,L)
202     di = sym(zeros(2,1));
203     gyi = gy(fz,my,cy,L);
204     [fzInt1,~,~] = Int(fz,L);
205     [gyInt1,~,~] = Int(gyi,L);
206     di(1) = fzInt1;
207     di(2) = cy*gyInt1;
208 end
209
210 function Qi = Q(cy,ks,L)
211     muy = mu(ks,L,cy);

```

```

212     Qi = sym(zeros(2,2));
213     syms x
214     Qi(1,1) = muy*(L-x)*(L^2+L*x-2*x^2+12/ks*cy);
215     Qi(2,2) = muy*(L-x)*(L^2-3*L*x+12/ks*cy);
216     Qi(1,2) = -muy*x*(L-x)*(L^2-L*x+6/ks*cy);
217     Qi(2,1) = 6*muy*x*(L-x);
218 end
219
220 function Ri = R(cy,ks,L)
221     muy = mu(ks,L,cy);
222     Ri = sym(zeros(2,2));
223     syms x
224     Ri(1,1) = muy*x*(3*L*x-2*x^2+12/ks*cy);
225     Ri(2,2) = muy*x*(-2*L^2+3*L*x+12/ks*cy);
226     Ri(1,2) = muy*x*(L-x)*(L*x+6/ks*cy);
227     Ri(2,1) = -6*muy*x*(L-x);
228 end

```

```

1  function [J,urz,uphiy,vz,vy] = OptControl(eps)
2
3  % Optcontrol: Solves an optimization problem
4  % of the Reduced Timoshenko beam
5  % with boundary conditions
6  % for a given epsilon value.
7  % Input:
8  % - eps: Value of epsilon. Default is 0.5.
9  % Output:
10 % - J: Cost function value.
11 % - urz, uphiy, vz, vy: Solution vectors.
12
13 % Default value for epsilon if not provided
14 if nargin == 0
15     eps = 0.5;
16 end
17
18 % Load data files
19 load("u_data.mat");
20 load("LAMBDA.mat");
21 load("fz.mat");
22 load("my.mat");
23
24 % material coefficient
25 ksv = 688500000;
26 cyv = 1749300;
27
28 u0 = zeros(6,1);
29
30 %% ===== Setting =====
31 eps0 = 0.5;
32 epsmax = 1;

```

```

33
34 % The length of each edge.
35 L = Len;
36 hmax = 0.01;
37
38 l = L - epsmax;
39 N0l = ceil(l/hmax);
40 x0l = linspace(0,l,N0l+1);
41
42 % number of basis functions
43 Nfy = 2*(N0l+1);
44
45 %% FEM matrices
46 [A1,A2,A3,A4] = graph1D.matirx_Hermite(N0l,x0l);
47
48 AA = cell(4,4);
49 for i = 1 : 4
50     for j = 1 : 4
51         AA{i,j} = deal(zeros(Nfy,Nfy));
52     end
53 end
54
55 AA{1,1} = A4;
56 AA{2,2} = A4;
57 AA{3,3} = -A4;
58 AA{4,4} = -A4;
59 AA{1,3} = ksv*A1;
60 AA{1,4} = ksv*A2;
61 AA{2,3} = ksv*A3;
62 AA{2,4} = ksv*A4+cyv*A1;
63 AA{3,1} = ksv*A1;
64 AA{3,2} = ksv*A2;
65 AA{4,1} = ksv*A3;
66 AA{4,2} = ksv*A4+cyv*A1;
67
68 AA = cell2mat(AA);
69
70 A = blkdiag(AA,AA,AA);
71
72 Nf = [repmat(Nfy,1,12)];
73 index = cumsum(Nf);
74 index = [0 index];
75 for i = 1 : 3
76     % index of [y(0),y(L),y'(0),y'(L)];
77     urz_bdy{i} = index(4*i-3)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
78     uphiy_bdy{i} = index(4*i-2)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
79     % index of [p(0),p(L),p'(0),p'(L)];
80     prz_bdy{i} = index(4*i-1)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
81     pphiy_bdy{i} = index(4*i)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
82 end
83 syms epsVal ks cy

```

```

84 Lval = subs(Lambda,[epsVal,ks,cy],[eps,ksv,cyv]);
85
86 A(urz_bdy{1}(2),prz_bdy{1}(2)) = A(urz_bdy{1}(2),prz_bdy{1}(2)) - ...
    Lval(1,1);
87 A(urz_bdy{1}(2),pphiy_bdy{1}(2)) = A(urz_bdy{1}(2),pphiy_bdy{1}(2)) - ...
    Lval(2,1);
88 A(urz_bdy{1}(2),prz_bdy{2}(2)) = A(urz_bdy{1}(2),prz_bdy{2}(2)) - ...
    Lval(3,1);
89 A(urz_bdy{1}(2),pphiy_bdy{2}(2)) = A(urz_bdy{1}(2),pphiy_bdy{2}(2)) - ...
    Lval(4,1);
90 A(urz_bdy{1}(2),prz_bdy{3}(2)) = A(urz_bdy{1}(2),prz_bdy{3}(2)) - ...
    Lval(5,1);
91 A(urz_bdy{1}(2),pphiy_bdy{3}(2)) = A(urz_bdy{1}(2),pphiy_bdy{3}(2)) - ...
    Lval(6,1);
92
93 A(uphiy_bdy{1}(2),prz_bdy{1}(2)) = A(uphiy_bdy{1}(2),prz_bdy{1}(2)) - ...
    Lval(1,2);
94 A(uphiy_bdy{1}(2),pphiy_bdy{1}(2)) = A(uphiy_bdy{1}(2),pphiy_bdy{1}(2)) ...
    - Lval(2,2);
95 A(uphiy_bdy{1}(2),prz_bdy{2}(2)) = A(uphiy_bdy{1}(2),prz_bdy{2}(2)) - ...
    Lval(3,2);
96 A(uphiy_bdy{1}(2),pphiy_bdy{2}(2)) = A(uphiy_bdy{1}(2),pphiy_bdy{2}(2)) ...
    - Lval(4,2);
97 A(uphiy_bdy{1}(2),prz_bdy{3}(2)) = A(uphiy_bdy{1}(2),prz_bdy{3}(2)) - ...
    Lval(5,2);
98 A(uphiy_bdy{1}(2),pphiy_bdy{3}(2)) = A(uphiy_bdy{1}(2),pphiy_bdy{3}(2)) ...
    - Lval(6,2);
99
100 A(urz_bdy{2}(2),prz_bdy{1}(2)) = A(urz_bdy{2}(2),prz_bdy{1}(2)) - ...
    Lval(1,3);
101 A(urz_bdy{2}(2),pphiy_bdy{1}(2)) = A(urz_bdy{2}(2),pphiy_bdy{1}(2)) - ...
    Lval(2,3);
102 A(urz_bdy{2}(2),prz_bdy{2}(2)) = A(urz_bdy{2}(2),prz_bdy{2}(2)) - ...
    Lval(3,3);
103 A(urz_bdy{2}(2),pphiy_bdy{2}(2)) = A(urz_bdy{2}(2),pphiy_bdy{2}(2)) - ...
    Lval(4,3);
104 A(urz_bdy{2}(2),prz_bdy{3}(2)) = A(urz_bdy{2}(2),prz_bdy{3}(2)) - ...
    Lval(5,3);
105 A(urz_bdy{2}(2),pphiy_bdy{3}(2)) = A(urz_bdy{2}(2),pphiy_bdy{3}(2)) - ...
    Lval(6,3);
106
107 A(uphiy_bdy{2}(2),prz_bdy{1}(2)) = A(uphiy_bdy{2}(2),prz_bdy{1}(2)) - ...
    Lval(1,4);
108 A(uphiy_bdy{2}(2),pphiy_bdy{1}(2)) = A(uphiy_bdy{2}(2),pphiy_bdy{1}(2)) ...
    - Lval(2,4);
109 A(uphiy_bdy{2}(2),prz_bdy{2}(2)) = A(uphiy_bdy{2}(2),prz_bdy{2}(2)) - ...
    Lval(3,4);
110 A(uphiy_bdy{2}(2),pphiy_bdy{2}(2)) = A(uphiy_bdy{2}(2),pphiy_bdy{2}(2)) ...
    - Lval(4,4);
111 A(uphiy_bdy{2}(2),prz_bdy{3}(2)) = A(uphiy_bdy{2}(2),prz_bdy{3}(2)) - ...
    Lval(5,4);

```

```

112 A(uphiy_bdy{2}(2),pphiy_bdy{3}(2)) = A(uphiy_bdy{2}(2),pphiy_bdy{3}(2)) ...
    - Lval(6,4);
113
114 A(urz_bdy{3}(2),prz_bdy{1}(2)) = A(urz_bdy{3}(2),prz_bdy{1}(2)) - ...
    Lval(1,5);
115 A(urz_bdy{3}(2),pphiy_bdy{1}(2)) = A(urz_bdy{3}(2),pphiy_bdy{1}(2)) - ...
    Lval(2,5);
116 A(urz_bdy{3}(2),prz_bdy{2}(2)) = A(urz_bdy{3}(2),prz_bdy{2}(2)) - ...
    Lval(3,5);
117 A(urz_bdy{3}(2),pphiy_bdy{2}(2)) = A(urz_bdy{3}(2),pphiy_bdy{2}(2)) - ...
    Lval(4,5);
118 A(urz_bdy{3}(2),prz_bdy{3}(2)) = A(urz_bdy{3}(2),prz_bdy{3}(2)) - ...
    Lval(5,5);
119 A(urz_bdy{3}(2),pphiy_bdy{3}(2)) = A(urz_bdy{3}(2),pphiy_bdy{3}(2)) - ...
    Lval(6,5);
120
121 A(uphiy_bdy{3}(2),prz_bdy{1}(2)) = A(uphiy_bdy{3}(2),prz_bdy{1}(2)) - ...
    Lval(1,6);
122 A(uphiy_bdy{3}(2),pphiy_bdy{1}(2)) = A(uphiy_bdy{3}(2),pphiy_bdy{1}(2)) ...
    - Lval(2,6);
123 A(uphiy_bdy{3}(2),prz_bdy{2}(2)) = A(uphiy_bdy{3}(2),prz_bdy{2}(2)) - ...
    Lval(3,6);
124 A(uphiy_bdy{3}(2),pphiy_bdy{2}(2)) = A(uphiy_bdy{3}(2),pphiy_bdy{2}(2)) ...
    - Lval(4,6);
125 A(uphiy_bdy{3}(2),prz_bdy{3}(2)) = A(uphiy_bdy{3}(2),prz_bdy{3}(2)) - ...
    Lval(5,6);
126 A(uphiy_bdy{3}(2),pphiy_bdy{3}(2)) = A(uphiy_bdy{3}(2),pphiy_bdy{3}(2)) ...
    - Lval(6,6);
127
128
129 A(prz_bdy{1}(2),urz_bdy{1}(2)) = A(prz_bdy{1}(2),urz_bdy{1}(2)) - ...
    Lval(1,1);
130 A(prz_bdy{1}(2),uphiy_bdy{1}(2)) = A(prz_bdy{1}(2),uphiy_bdy{1}(2)) - ...
    Lval(2,1);
131 A(prz_bdy{1}(2),urz_bdy{2}(2)) = A(prz_bdy{1}(2),urz_bdy{2}(2)) - ...
    Lval(3,1);
132 A(prz_bdy{1}(2),uphiy_bdy{2}(2)) = A(prz_bdy{1}(2),uphiy_bdy{2}(2)) - ...
    Lval(4,1);
133 A(prz_bdy{1}(2),urz_bdy{3}(2)) = A(prz_bdy{1}(2),urz_bdy{3}(2)) - ...
    Lval(5,1);
134 A(prz_bdy{1}(2),uphiy_bdy{3}(2)) = A(prz_bdy{1}(2),uphiy_bdy{3}(2)) - ...
    Lval(6,1);
135
136 A(pphiy_bdy{1}(2),urz_bdy{1}(2)) = A(pphiy_bdy{1}(2),urz_bdy{1}(2)) - ...
    Lval(1,2);
137 A(pphiy_bdy{1}(2),uphiy_bdy{1}(2)) = A(pphiy_bdy{1}(2),uphiy_bdy{1}(2)) ...
    - Lval(2,2);
138 A(pphiy_bdy{1}(2),urz_bdy{2}(2)) = A(pphiy_bdy{1}(2),urz_bdy{2}(2)) - ...
    Lval(3,2);
139 A(pphiy_bdy{1}(2),uphiy_bdy{2}(2)) = A(pphiy_bdy{1}(2),uphiy_bdy{2}(2)) ...
    - Lval(4,2);

```

```

140 A(pphiy_bdy{1}(2),urz_bdy{3}(2)) = A(pphiy_bdy{1}(2),urz_bdy{3}(2)) - ...
    Lval(5,2);
141 A(pphiy_bdy{1}(2),uphiy_bdy{3}(2)) = A(pphiy_bdy{1}(2),uphiy_bdy{3}(2)) ...
    - Lval(6,2);
142
143 A(prz_bdy{2}(2),urz_bdy{1}(2)) = A(prz_bdy{2}(2),urz_bdy{1}(2)) - ...
    Lval(1,3);
144 A(prz_bdy{2}(2),uphiy_bdy{1}(2)) = A(prz_bdy{2}(2),uphiy_bdy{1}(2)) - ...
    Lval(2,3);
145 A(prz_bdy{2}(2),urz_bdy{2}(2)) = A(prz_bdy{2}(2),urz_bdy{2}(2)) - ...
    Lval(3,3);
146 A(prz_bdy{2}(2),uphiy_bdy{2}(2)) = A(prz_bdy{2}(2),uphiy_bdy{2}(2)) - ...
    Lval(4,3);
147 A(prz_bdy{2}(2),urz_bdy{3}(2)) = A(prz_bdy{2}(2),urz_bdy{3}(2)) - ...
    Lval(5,3);
148 A(prz_bdy{2}(2),uphiy_bdy{3}(2)) = A(prz_bdy{2}(2),uphiy_bdy{3}(2)) - ...
    Lval(6,3);
149
150 A(pphiy_bdy{2}(2),urz_bdy{1}(2)) = A(pphiy_bdy{2}(2),urz_bdy{1}(2)) - ...
    Lval(1,4);
151 A(pphiy_bdy{2}(2),uphiy_bdy{1}(2)) = A(pphiy_bdy{2}(2),uphiy_bdy{1}(2)) ...
    - Lval(2,4);
152 A(pphiy_bdy{2}(2),urz_bdy{2}(2)) = A(pphiy_bdy{2}(2),urz_bdy{2}(2)) - ...
    Lval(3,4);
153 A(pphiy_bdy{2}(2),uphiy_bdy{2}(2)) = A(pphiy_bdy{2}(2),uphiy_bdy{2}(2)) ...
    - Lval(4,4);
154 A(pphiy_bdy{2}(2),urz_bdy{3}(2)) = A(pphiy_bdy{2}(2),urz_bdy{3}(2)) - ...
    Lval(5,4);
155 A(pphiy_bdy{2}(2),uphiy_bdy{3}(2)) = A(pphiy_bdy{2}(2),uphiy_bdy{3}(2)) ...
    - Lval(6,4);
156
157 A(prz_bdy{3}(2),urz_bdy{1}(2)) = A(prz_bdy{3}(2),urz_bdy{1}(2)) - ...
    Lval(1,5);
158 A(prz_bdy{3}(2),uphiy_bdy{1}(2)) = A(prz_bdy{3}(2),uphiy_bdy{1}(2)) - ...
    Lval(2,5);
159 A(prz_bdy{3}(2),urz_bdy{2}(2)) = A(prz_bdy{3}(2),urz_bdy{2}(2)) - ...
    Lval(3,5);
160 A(prz_bdy{3}(2),uphiy_bdy{2}(2)) = A(prz_bdy{3}(2),uphiy_bdy{2}(2)) - ...
    Lval(4,5);
161 A(prz_bdy{3}(2),urz_bdy{3}(2)) = A(prz_bdy{3}(2),urz_bdy{3}(2)) - ...
    Lval(5,5);
162 A(prz_bdy{3}(2),uphiy_bdy{3}(2)) = A(prz_bdy{3}(2),uphiy_bdy{3}(2)) - ...
    Lval(6,5);
163
164 A(pphiy_bdy{3}(2),urz_bdy{1}(2)) = A(pphiy_bdy{3}(2),urz_bdy{1}(2)) - ...
    Lval(1,6);
165 A(pphiy_bdy{3}(2),uphiy_bdy{1}(2)) = A(pphiy_bdy{3}(2),uphiy_bdy{1}(2)) ...
    - Lval(2,6);
166 A(pphiy_bdy{3}(2),urz_bdy{2}(2)) = A(pphiy_bdy{3}(2),urz_bdy{2}(2)) - ...
    Lval(3,6);
167 A(pphiy_bdy{3}(2),uphiy_bdy{2}(2)) = A(pphiy_bdy{3}(2),uphiy_bdy{2}(2)) ...

```

```

- Lval(4,6);
168 A(pphiy_bdy{3}(2),urz_bdy{3}(2)) = A(pphiy_bdy{3}(2),urz_bdy{3}(2)) - ...
    Lval(5,6);
169 A(pphiy_bdy{3}(2),uphiy_bdy{3}(2)) = A(pphiy_bdy{3}(2),uphiy_bdy{3}(2)) ...
    - Lval(6,6);
170
171 syms x
172 for i = 1 : 3
173     z{i} = u{i};
174     zNum{i} = double(subs(z{i},x01));
175     dz{i} = diff(z{i},x);
176     dzNum{i} = double(subs(dz{i},x01));
177     dzNum{i}(:,end) = double(subs(dz{i}, x, 0.999999));
178     dzNum{i}(:,1) = double(subs(dz{i}, x, 0.000001));
179
180     dfz{i} = diff(fz(i),x);
181     dfzNum{i} = double(subs(dfz{i},x01));
182     dfzNum{i}(:,end) = double(subs(dfz{i}, x, 0.999999));
183     dfzNum{i}(:,1) = double(subs(dfz{i}, x, 0.000001));
184
185     dmy{i} = diff(my(i),x);
186     dmyNum{i} = double(subs(dmy{i},x01));
187     dmyNum{i}(:,end) = double(subs(dmy{i}, x, 0.999999));
188     dmyNum{i}(:,1) = double(subs(dmy{i}, x, 0.000001));
189 end
190
191 b = zeros(size(A,1),1);
192
193 b(1:Nfy) = A4*[zNum{1}(1,:) dzNum{1}(1,:)*hmax]';
194 b(Nfy+1:2*Nfy) = A4*[zNum{1}(2,:) dzNum{1}(2,:)*hmax]';
195 b(2*Nfy+1:3*Nfy) = A4*[double(subs(fz(1),x01)) dfzNum{1}*hmax]';
196 b(3*Nfy+1:4*Nfy) = A4*[double(subs(my(2),x01)) dmyNum{2}*hmax]';
197
198 b(4*Nfy+1:5*Nfy) = A4*[zNum{2}(1,:) dzNum{2}(1,:)*hmax]';
199 b(5*Nfy+1:6*Nfy) = A4*[zNum{2}(2,:) dzNum{2}(2,:)*hmax]';
200 b(6*Nfy+1:7*Nfy) = A4*[double(subs(fz(2),x01)) dfzNum{2}*hmax]';
201 b(7*Nfy+1:8*Nfy) = A4*[double(subs(my(2),x01)) dmyNum{2}*hmax]';
202
203 b(8*Nfy+1:9*Nfy) = A4*[zNum{3}(1,:) dzNum{3}(1,:)*hmax]';
204 b(9*Nfy+1:10*Nfy) = A4*[zNum{3}(2,:) dzNum{3}(2,:)*hmax]';
205 b(10*Nfy+1:11*Nfy) = A4*[double(subs(fz(3),x01)) dfzNum{3}*hmax]';
206 b(11*Nfy+1:12*Nfy) = A4*[double(subs(my(3),x01)) dmyNum{3}*hmax]';
207
208 % boundary condition
209 for i = 1 : 3
210     A(urz_bdy{i}(1),:) = 0;
211     A(urz_bdy{i}(1),urz_bdy{i}(1)) = 1;
212     b(urz_bdy{i}(1)) = u0(i);
213
214     A(uphiy_bdy{i}(1),:) = 0;
215     A(uphiy_bdy{i}(1),uphiy_bdy{i}(1)) = 1;

```

```

216     b(uphiy_bdy{i}(1)) = u0(i);
217
218     A(prz_bdy{i}(1), :) = 0;
219     A(prz_bdy{i}(1), prz_bdy{i}(1)) = 1;
220     b(prz_bdy{i}(1)) = u0(i);
221
222     A(pphiy_bdy{i}(1), :) = 0;
223     A(pphiy_bdy{i}(1), pphiy_bdy{i}(1)) = 1;
224     b(pphiy_bdy{i}(1)) = u0(i);
225 end
226
227 solu = A\b;
228
229 for i = 1 : 3
230     urz{i} = solu(urz_bdy{i}(1):urz_bdy{i}(2));
231     uphiy{i} = solu(uphiy_bdy{i}(1):uphiy_bdy{i}(2));
232     prz{i} = solu(prz_bdy{i}(1):prz_bdy{i}(2));
233     pphiy{i} = solu(pphiy_bdy{i}(1):pphiy_bdy{i}(2));
234     vz{i} = prz{i} + subs(fz(i), x0l)';
235     vy{i} = pphiy{i} + subs(my(i), x0l)';
236 end
237
238 J = 0;
239 for i = 1 : 3
240     tmp = norm(urz{i}-double(subs(z{i}(1), x0l))')^2*hmax;
241     J = J + tmp;
242     tmp = norm(uphiy{i}-double(subs(z{i}(2), x0l))')^2*hmax;
243     J = J + tmp;
244     tmp = norm(prz{i})^2*hmax;
245     J = J + tmp;
246     tmp = norm(pphiy{i})^2*hmax;
247     J = J + tmp;
248 end
249 J = J/2;
250
251 end

```

```

1  % This is to get the exact solution of the 3D model
2
3  load("ks.mat")
4  load("kx.mat")
5  load("cx.mat")
6  load("cy.mat")
7  load("cz.mat")
8
9  % Define lengths
10 L = [2-epsVal, 2-epsVal, 2-epsVal, epsVal, epsVal, epsVal];
11
12 % Define piecewise distributed force
13 syms x;

```

```

14 f = piecewise(x <= 1.00000, 2e10, x > 1.00000 & x <= L(2), 0);
15
16 % Define local forces and moments
17 fx = [0,-f,0,0,0,0];
18 fy = [0,0,0,0,0,0];
19 fz = [0,0,0,0,0,0];
20 mx = [0,0,0,0,0,0];
21 my = [0,0,0,0,0,0];
22 mz = [0,0,0,0,0,0];
23
24 % Define local displacement and rotation
25 rx = [0,0,0,0,0,0];
26 ry = [0,0,0,0,0,0];
27 rz = [0,0,0,0,0,0];
28 phix = [0,0,0,0,0,0];
29 phiy = [0,0,0,0,0,0];
30 phiz = [0,0,0,0,0,0];
31
32 for i = 1 : 6
33     q{i} = [fx(i);fy(i);fz(i);mx(i);my(i);mz(i)];
34     u0{i} = [rx(i);ry(i);rz(i);phix(i);phiy(i);phiz(i)];
35 end
36
37 % Define rotation angles
38 theta = [-1/6*pi,pi/2,-5/6*pi,-pi,-pi/3,pi/3];
39
40 % Compute matrices based on rotation angles
41 [DFe,DPe,DMe,Se] = deal(cell(6,1));
42 for i = 1 : 6
43     DFe{i} = simplify(E(theta(i))*DF(kx,ks,L(i),cy,cz)*E(theta(i)).');
44     DPe{i} = simplify(E(theta(i))*DP(ks,L(i),cx,cy,cz)*E(theta(i)).');
45     DMe{i} = simplify(E(theta(i))*DM(ks,L(i),cx,cy,cz)*E(theta(i)).');
46     Se{i} = simplify(E(theta(i))*S(ks,L(i),cy,cz)*E(theta(i)).');
47 end
48
49 for i = 1 : 6
50     D1E{i} = -[DFe{i} -Se{i}.';-Se{i} DPe{i}];
51     D2E{i} = [DFe{i} Se{i}.';-Se{i} DMe{i}];
52     D3E{i} = -[DFe{i} -Se{i}.'; Se{i} DMe{i}];
53     D4E{i} = [DFe{i} Se{i}.'; Se{i} DPe{i}];
54 end
55
56 % Compute forces, moments, and displacements based on rotation angles
57 for i = 1 : 6
58     dE{i} = blkdiag(E(theta(i)),E(theta(i)))
59             *d(fx(i),fy(i),fz(i),mx(i),my(i),mz(i),cy,cz,L(i));
60     wE{i} = blkdiag(E(theta(i)),E(theta(i)))
61             *w(fx(i),fy(i),fz(i),mx(i),my(i),mz(i),kx,ks,cx,cy,cz,L(i));
62     RE{i} = blkdiag(E(theta(i)),E(theta(i)))
63             *R(cy,cz,ks,L(i))*blkdiag(E(theta(i))',E(theta(i))');
64     QE{i} = blkdiag(E(theta(i)),E(theta(i)))

```

```

65         *Q(cy,cz,ks,L(i))*blkdiag(E(theta(i))',E(theta(i))');
66 end
67
68 % Compute the global A
69 A = [D4E{1}+D4E{4}-D1E{5}, -D2E{5}, D3E{4};
70      D3E{5}, D4E{2}+D4E{5}-D1E{6}, -D2E{6};
71      -D2E{4}, D3E{6}, D4E{3}+D4E{6}-D1E{4}]; %symmertic
72
73 b = sym(zeros(18,1));
74 b(1:6,1) = - q{1} - D3E{1}*u0{1} ...
75            - D4E{1}*subs(wE{1},L(1)) - D4E{4}*subs(wE{4},L(4)) + ...
76            D2E{5}*subs(wE{5},L(5)) ...
77            + subs(dE{1},L(1)) + subs(dE{4},L(4));
78 b(7:12,1) = - q{2} - D3E{2}*u0{2} ...
79            - D4E{2}*subs(wE{2},L(2)) - D4E{5}*subs(wE{5},L(5)) + ...
80            D2E{6}*subs(wE{6},L(6)) ...
81            + subs(dE{2},L(2)) + subs(dE{5},L(5));
82 b(13:18,1) = - q{3} - D3E{3}*u0{3} ...
83            - D4E{3}*subs(wE{3},L(3)) - D4E{6}*subs(wE{6},L(6)) + ...
84            D2E{4}*subs(wE{4},L(4)) ...
85            + subs(dE{3},L(3)) + subs(dE{6},L(6));
86
87 solu = A\b;
88
89 % Extract local displacements
90 uL{1} = subs(solu(1:6),L(1));
91 uL{2} = subs(solu(7:12),L(2));
92 uL{3} = subs(solu(13:18),L(3));
93
94 u0{4} = uL{3};
95 u0{5} = uL{1};
96 u0{6} = uL{2};
97
98 uL{4} = uL{1};
99 uL{5} = uL{2};
100 uL{6} = uL{3};
101
102 % Here u is Global
103 for i = 1 : 6
104     u{i} = QE{i}*u0{i} + RE{i}*(subs(uL{i},L(i))+subs(wE{i},L(i)))-wE{i};
105 end
106
107 for i = 1 : 6
108     ulocal{i} = blkdiag(E(theta(i)),E(theta(i)))'*u{i};
109 end
110
111 % save('uglo.mat', 'u');
112 % save('uloc.mat', 'ulocal');
113 % save('q.mat', 'q');
114 % close all

```

```

113 function DFi = DF(kx,ks,L,cy,cz)
114     [muy,muz] = mu(ks,L,cy,cz);
115     DFi = sym(zeros(3,3));
116     DFi(1,1) = 1/L*kx;
117     DFi(2,2) = 12*cz*muz;
118     DFi(3,3) = 12*cy*muy;
119 end
120
121 function DPi = DP(ks,L,cx,cy,cz)
122     [muy,muz] = mu(ks,L,cy,cz);
123     DPi = sym(zeros(3,3));
124     DPi(1,1) = 1/L*cx;
125     DPi(2,2) = cy*(1/L+3*muy*L^2);
126     DPi(3,3) = cz*(1/L+3*muz*L^2);
127 end
128
129 function DMi = DM(ks,L,cx,cy,cz)
130     [muy,muz] = mu(ks,L,cy,cz);
131     DMi = sym(zeros(3,3));
132     DMi(1,1) = 1/L*cx;
133     DMi(2,2) = cy*(1/L-3*muy*L^2);
134     DMi(3,3) = cz*(1/L-3*muz*L^2);
135 end
136
137 function Si = S(ks,L,cy,cz)
138     [muy,muz] = mu(ks,L,cy,cz);
139     Si = sym(zeros(3,3));
140     Si(2,3) = 6*cy*muy*L;
141     Si(3,2) = -6*cz*muz*L;
142 end
143
144 function [muy,muz] = mu(ks,L,cy,cz)
145     muy = ks/(ks*L^3+12*cy*L);
146     muz = ks/(ks*L^3+12*cz*L);
147 end
148
149 function Eil = E(t)
150     Eil = [cos(t) sin(t) 0;
151           -sin(t) cos(t) 0;
152           0 0 1];
153 end
154
155 function [fInt,fInt2,fInt3] = Int(f,L)
156     syms x;
157     fInt = int(f,x,0,x);
158     fInt2 = int(fInt,x,0,x);
159     fInt3 = int(fInt2,x,0,x);
160 end
161
162 function [gyi,gzi] = gy(fy,fz,my,mz,cy,cz,L)
163     [fzInt1,~,~] = Int(fz,L);

```

```

164     [fyInt1,~,~] = Int(fy,L);
165     gyi = 1/cy*(my+fzInt1);
166     gzi = 1/cz*(mz-fyInt1);
167 end
168
169 function wi = w(fx,fy,fz,mx,my,mz,kx,ks,cx,cy,cz,L)
170     wi = sym(zeros(6,1));
171     [gyi,gzi] = gy(fy,fz,my,mz,cy,cz,L);
172
173     [~,fzInt2,~] = Int(fz,L);
174     [~,fxInt2,~] = Int(fx,L);
175     [~,fyInt2,~] = Int(fy,L);
176
177     [~,gyInt2,gyInt3] = Int(gyi,L);
178     [~,gzInt2,gzInt3] = Int(gzi,L);
179
180     [~,mxInt2,~] = Int(mx,L);
181
182     wi(3) = 1/ks*fzInt2 - gyInt3;
183     wi(5) = gyInt2;
184
185     wi(1) = 1/kx*fxInt2;
186     wi(2) = 1/ks*fyInt2 + gzInt3;
187     wi(4) = 1/cx*(mxInt2);
188     wi(6) = gzInt2;
189 end
190
191 function di = d(fx,fy,fz,mx,my,mz,cy,cz,L)
192     di = sym(zeros(2,1));
193     [gyi,gzi] = gy(fy,fz,my,mz,cy,cz,L);
194     [fxInt1,~,~] = Int(fx,L);
195     [fzInt1,~,~] = Int(fz,L);
196     [fyInt1,~,~] = Int(fy,L);
197     [mxInt1,~,~] = Int(mx,L);
198
199     [gyInt1,~,~] = Int(gyi,L);
200     [gzInt1,~,~] = Int(gzi,L);
201     di(3) = fzInt1;
202     di(5) = cy*gyInt1;
203
204     di(1) = fxInt1;
205     di(2) = fyInt1;
206     di(4) = mxInt1;
207     di(6) = cz*gzInt1;
208
209 end
210
211 function Qi = Q(cy,cz,ks,L)
212     [muy,muz] = mu(ks,L,cy,cz);
213     Qi = sym(zeros(6,6));
214     syms x

```

```

215     Qi(3,3) = muy*(L-x)*(L^2+L*x-2*x^2+12/ks*cy);
216     Qi(5,5) = muy*(L-x)*(L^2-3*L*x+12/ks*cy);
217     Qi(3,5) = -muy*x*(L-x)*(L^2-L*x+6/ks*cy);
218     Qi(5,3) = 6*muy*x*(L-x);
219
220     Qi(1,1) = 1 - L^-1*x;
221     Qi(2,2) = muz*(L-x)*(L^2+L*x-2*x^2+12*ks^-1*czy);
222     Qi(4,4) = 1 - L^-1*x;
223     Qi(6,6) = muz*(L-x)*(L^2-3*L*x+12/ks*czy);
224     Qi(2,6) = muz*x*(L-x)*(L^2-L*x+6*ks^-1*czy);
225     Qi(6,2) = -6*muz*x*(L-x);
226
227 end
228
229 function Ri = R(cy,cz,ks,L)
230     [muy,muz] = mu(ks,L,cy,cz);
231     Ri = sym(zeros(6,6));
232     syms x
233     Ri(3,3) = muy*x*(3*L*x-2*x^2+12/ks*cy);
234     Ri(5,5) = muy*x*(-2*L^2+3*L*x+12/ks*cy);
235     Ri(3,5) = muy*x*(L-x)*(L*x+6/ks*cy);
236     Ri(5,3) = -6*muy*x*(L-x);
237
238     Ri(1,1) = 1/L*x;
239     Ri(2,2) = muz*x*(3*L*x-2*x^2+12/ks*czy);
240     Ri(4,4) = 1/L*x;
241     Ri(6,6) = muz*x*(-2*L^2+3*L*x+12/ks*czy);
242     Ri(2,6) = -muz*x*(L-x)*(L*x+6/ks*czy);
243     Ri(6,2) = 6*muz*x*(L-x);
244 end

```

```

1 function J = OptControlFull(epsVal)
2 % Function to calculate the optimal control based on certain epsilon ...
   value
3
4 % If epsilon value is not provided, set it to 0.5
5 if nargin == 0
6     epsVal = 0.5;
7 end
8
9 % Load necessary data
10 load("ks.mat")
11 load("kx.mat")
12 load("cx.mat")
13 load("cy.mat")
14 load("cz.mat")
15
16 load("uloc.mat");
17 load("q.mat");
18

```

```

19 % Calculate operator lambda value
20 Lval = Val_lambdaFullModel(epsVal);
21
22 u0 = zeros(6,1);
23
24 %% ===== Setting up parameters =====
25 epsmax = 1;
26
27 theta = [-1/6*pi,pi/2,-5/6*pi,-pi,-pi/3,pi/3];
28
29 % Define length and discretization
30 L = 2;
31 hmax = 0.02;
32
33 l = L-epsmax;
34 N0l = ceil(l/hmax);
35 x0l = linspace(0,L-epsmax,N0l+1);
36
37 % number of basis functions
38 Nfy = 2*(N0l+1);
39
40 %% FEM matrices
41 [A1,A2,A3,A4] = graph1D.matirx_Hermite(N0l,x0l);
42
43 AA = cell(12,12);
44 for i = 1 : 12
45     for j = 1 : 12
46         AA{i,j} = deal(zeros(Nfy,Nfy));
47     end
48 end
49
50 for i = 1 : 6
51     AA{i,i} = A4;
52 end
53
54 for i = 7 : 12
55     AA{i,i} = -A4;
56 end
57
58 AA{3,3+6} = ks*A1;
59 AA{3,5+6} = ks*A2;
60 AA{5,3+6} = ks*A3;
61 AA{5,5+6} = ks*A4+cy*A1;
62
63 AA{1,1+6} = kx*A1;
64 AA{2,2+6} = ks*A1;
65 AA{4,4+6} = cx*A1;
66 AA{6,6+6} = ks*A4+cz*A1;
67 AA{2,6+6} = -ks*A2;
68 AA{6,2+6} = -ks*A3;
69

```

```

70 AA{3+6,3} = ks*A1;
71 AA{3+6,5} = ks*A2;
72 AA{5+6,3} = ks*A3;
73 AA{5+6,5} = ks*A4+cy*A1;
74
75 AA{1+6,1} = kx*A1;
76 AA{2+6,2} = ks*A1;
77 AA{4+6,4} = cx*A1;
78 AA{6+6,6} = ks*A4+cz*A1;
79 AA{2+6,6} = -ks*A2;
80 AA{6+6,2} = -ks*A3;
81
82 AA = cell2mat(AA);
83
84 Aold = blkdiag(AA,AA,AA);
85 A = Aold;
86
87 % Set up boundary indices
88 Nf = [repmat(Nfy,1,36)];
89 index = cumsum(Nf);
90 index = [0 index];
91 for i = 1 : 3
92     % index of [y(0),y(L),y'(0),y'(L)];
93     urx_bdy{i} = index(12*i-11)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
94     ury_bdy{i} = index(12*i-10)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
95     urz_bdy{i} = index(12*i-9)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
96     uphix_bdy{i} = index(12*i-8)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
97     uphiy_bdy{i} = index(12*i-7)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
98     uphiz_bdy{i} = index(12*i-6)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
99     % index of [p(0),p(L),p'(0),p'(L)];
100    prx_bdy{i} = index(12*i-5)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
101    pry_bdy{i} = index(12*i-4)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
102    prz_bdy{i} = index(12*i-3)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
103    pphix_bdy{i} = index(12*i-2)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
104    pphiy_bdy{i} = index(12*i-1)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
105    pphiz_bdy{i} = index(12*i)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
106 end
107
108 % Rotate lambda
109 E3 = blkdiag(E(theta(1)),E(theta(1)), ...
110             E(theta(2)),E(theta(2)), ...
111             E(theta(3)),E(theta(3)));
112
113 Lval = E3'*Lval*E3;
114
115 % Update A matrix with lambda values at boundaries
116 for i = 1 : 3
117     A(urx_bdy{i}(2),prx_bdy{1}(2)) = A(urx_bdy{i}(2),prx_bdy{1}(2)) - ...
118         Lval(1,1+(i-1)*6);
119     A(ury_bdy{i}(2),pry_bdy{1}(2)) = A(ury_bdy{i}(2),pry_bdy{1}(2)) - ...
120         Lval(2,1+(i-1)*6);

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119  A(urx_bdy{i}(2),prz_bdy{1}(2)) = A(urx_bdy{i}(2),prz_bdy{1}(2)) - ...
      Lval(3,1+(i-1)*6);
120  A(urx_bdy{i}(2),pphix_bdy{1}(2)) = A(urx_bdy{i}(2),pphix_bdy{1}(2)) ...
      - Lval(4,1+(i-1)*6);
121  A(urx_bdy{i}(2),pphiy_bdy{1}(2)) = A(urx_bdy{i}(2),pphiy_bdy{1}(2)) ...
      - Lval(5,1+(i-1)*6);
122  A(urx_bdy{i}(2),pphiz_bdy{1}(2)) = A(urx_bdy{i}(2),pphiz_bdy{1}(2)) ...
      - Lval(6,1+(i-1)*6);
123  A(urx_bdy{i}(2),prx_bdy{2}(2)) = A(urx_bdy{i}(2),prx_bdy{2}(2)) - ...
      Lval(7,1+(i-1)*6);
124  A(urx_bdy{i}(2),pry_bdy{2}(2)) = A(urx_bdy{i}(2),pry_bdy{2}(2)) - ...
      Lval(8,1+(i-1)*6);
125  A(urx_bdy{i}(2),prz_bdy{2}(2)) = A(urx_bdy{i}(2),prz_bdy{2}(2)) - ...
      Lval(9,1+(i-1)*6);
126  A(urx_bdy{i}(2),pphix_bdy{2}(2)) = A(urx_bdy{i}(2),pphix_bdy{2}(2)) ...
      - Lval(10,1+(i-1)*6);
127  A(urx_bdy{i}(2),pphiy_bdy{2}(2)) = A(urx_bdy{i}(2),pphiy_bdy{2}(2)) ...
      - Lval(11,1+(i-1)*6);
128  A(urx_bdy{i}(2),pphiz_bdy{2}(2)) = A(urx_bdy{i}(2),pphiz_bdy{2}(2)) ...
      - Lval(12,1+(i-1)*6);
129  A(urx_bdy{i}(2),prx_bdy{3}(2)) = A(urx_bdy{i}(2),prx_bdy{3}(2)) - ...
      Lval(13,1+(i-1)*6);
130  A(urx_bdy{i}(2),pry_bdy{3}(2)) = A(urx_bdy{i}(2),pry_bdy{3}(2)) - ...
      Lval(14,1+(i-1)*6);
131  A(urx_bdy{i}(2),prz_bdy{3}(2)) = A(urx_bdy{i}(2),prz_bdy{3}(2)) - ...
      Lval(15,1+(i-1)*6);
132  A(urx_bdy{i}(2),pphix_bdy{3}(2)) = A(urx_bdy{i}(2),pphix_bdy{3}(2)) ...
      - Lval(16,1+(i-1)*6);
133  A(urx_bdy{i}(2),pphiy_bdy{3}(2)) = A(urx_bdy{i}(2),pphiy_bdy{3}(2)) ...
      - Lval(17,1+(i-1)*6);
134  A(urx_bdy{i}(2),pphiz_bdy{3}(2)) = A(urx_bdy{i}(2),pphiz_bdy{3}(2)) ...
      - Lval(18,1+(i-1)*6);
135
136  A(ury_bdy{i}(2),prx_bdy{1}(2)) = A(ury_bdy{i}(2),prx_bdy{1}(2)) - ...
      Lval(1,2+(i-1)*6);
137  A(ury_bdy{i}(2),pry_bdy{1}(2)) = A(ury_bdy{i}(2),pry_bdy{1}(2)) - ...
      Lval(2,2+(i-1)*6);
138  A(ury_bdy{i}(2),prz_bdy{1}(2)) = A(ury_bdy{i}(2),prz_bdy{1}(2)) - ...
      Lval(3,2+(i-1)*6);
139  A(ury_bdy{i}(2),pphix_bdy{1}(2)) = A(ury_bdy{i}(2),pphix_bdy{1}(2)) ...
      - Lval(4,2+(i-1)*6);
140  A(ury_bdy{i}(2),pphiy_bdy{1}(2)) = A(ury_bdy{i}(2),pphiy_bdy{1}(2)) ...
      - Lval(5,2+(i-1)*6);
141  A(ury_bdy{i}(2),pphiz_bdy{1}(2)) = A(ury_bdy{i}(2),pphiz_bdy{1}(2)) ...
      - Lval(6,2+(i-1)*6);
142  A(ury_bdy{i}(2),prx_bdy{2}(2)) = A(ury_bdy{i}(2),prx_bdy{2}(2)) - ...
      Lval(7,2+(i-1)*6);
143  A(ury_bdy{i}(2),pry_bdy{2}(2)) = A(ury_bdy{i}(2),pry_bdy{2}(2)) - ...
      Lval(8,2+(i-1)*6);
144  A(ury_bdy{i}(2),prz_bdy{2}(2)) = A(ury_bdy{i}(2),prz_bdy{2}(2)) - ...
      Lval(9,2+(i-1)*6);

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145  A(ury_bdy{i}(2),pphix_bdy{2}(2)) = A(ury_bdy{i}(2),pphix_bdy{2}(2)) ...
      - Lval(10,2+(i-1)*6);
146  A(ury_bdy{i}(2),pphiy_bdy{2}(2)) = A(ury_bdy{i}(2),pphiy_bdy{2}(2)) ...
      - Lval(11,2+(i-1)*6);
147  A(ury_bdy{i}(2),pphiz_bdy{2}(2)) = A(ury_bdy{i}(2),pphiz_bdy{2}(2)) ...
      - Lval(12,2+(i-1)*6);
148  A(ury_bdy{i}(2),prx_bdy{3}(2)) = A(ury_bdy{i}(2),prx_bdy{3}(2)) - ...
      Lval(13,2+(i-1)*6);
149  A(ury_bdy{i}(2),pry_bdy{3}(2)) = A(ury_bdy{i}(2),pry_bdy{3}(2)) - ...
      Lval(14,2+(i-1)*6);
150  A(ury_bdy{i}(2),prz_bdy{3}(2)) = A(ury_bdy{i}(2),prz_bdy{3}(2)) - ...
      Lval(15,2+(i-1)*6);
151  A(ury_bdy{i}(2),pphix_bdy{3}(2)) = A(ury_bdy{i}(2),pphix_bdy{3}(2)) ...
      - Lval(16,2+(i-1)*6);
152  A(ury_bdy{i}(2),pphiy_bdy{3}(2)) = A(ury_bdy{i}(2),pphiy_bdy{3}(2)) ...
      - Lval(17,2+(i-1)*6);
153  A(ury_bdy{i}(2),pphiz_bdy{3}(2)) = A(ury_bdy{i}(2),pphiz_bdy{3}(2)) ...
      - Lval(18,2+(i-1)*6);
154
155  A(urz_bdy{i}(2),prx_bdy{1}(2)) = A(urz_bdy{i}(2),prx_bdy{1}(2)) - ...
      Lval(1,3+(i-1)*6);
156  A(urz_bdy{i}(2),pry_bdy{1}(2)) = A(urz_bdy{i}(2),pry_bdy{1}(2)) - ...
      Lval(2,3+(i-1)*6);
157  A(urz_bdy{i}(2),prz_bdy{1}(2)) = A(urz_bdy{i}(2),prz_bdy{1}(2)) - ...
      Lval(3,3+(i-1)*6);
158  A(urz_bdy{i}(2),pphix_bdy{1}(2)) = A(urz_bdy{i}(2),pphix_bdy{1}(2)) ...
      - Lval(4,3+(i-1)*6);
159  A(urz_bdy{i}(2),pphiy_bdy{1}(2)) = A(urz_bdy{i}(2),pphiy_bdy{1}(2)) ...
      - Lval(5,3+(i-1)*6);
160  A(urz_bdy{i}(2),pphiz_bdy{1}(2)) = A(urz_bdy{i}(2),pphiz_bdy{1}(2)) ...
      - Lval(6,3+(i-1)*6);
161  A(urz_bdy{i}(2),prx_bdy{2}(2)) = A(urz_bdy{i}(2),prx_bdy{2}(2)) - ...
      Lval(7,3+(i-1)*6);
162  A(urz_bdy{i}(2),pry_bdy{2}(2)) = A(urz_bdy{i}(2),pry_bdy{2}(2)) - ...
      Lval(8,3+(i-1)*6);
163  A(urz_bdy{i}(2),prz_bdy{2}(2)) = A(urz_bdy{i}(2),prz_bdy{2}(2)) - ...
      Lval(9,3+(i-1)*6);
164  A(urz_bdy{i}(2),pphix_bdy{2}(2)) = A(urz_bdy{i}(2),pphix_bdy{2}(2)) ...
      - Lval(10,3+(i-1)*6);
165  A(urz_bdy{i}(2),pphiy_bdy{2}(2)) = A(urz_bdy{i}(2),pphiy_bdy{2}(2)) ...
      - Lval(11,3+(i-1)*6);
166  A(urz_bdy{i}(2),pphiz_bdy{2}(2)) = A(urz_bdy{i}(2),pphiz_bdy{2}(2)) ...
      - Lval(12,3+(i-1)*6);
167  A(urz_bdy{i}(2),prx_bdy{3}(2)) = A(urz_bdy{i}(2),prx_bdy{3}(2)) - ...
      Lval(13,3+(i-1)*6);
168  A(urz_bdy{i}(2),pry_bdy{3}(2)) = A(urz_bdy{i}(2),pry_bdy{3}(2)) - ...
      Lval(14,3+(i-1)*6);
169  A(urz_bdy{i}(2),prz_bdy{3}(2)) = A(urz_bdy{i}(2),prz_bdy{3}(2)) - ...
      Lval(15,3+(i-1)*6);
170  A(urz_bdy{i}(2),pphix_bdy{3}(2)) = A(urz_bdy{i}(2),pphix_bdy{3}(2)) ...
      - Lval(16,3+(i-1)*6);

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171  A(urz_bdy{i}(2),pphiy_bdy{3}(2)) = A(urz_bdy{i}(2),pphiy_bdy{3}(2)) ...
      - Lval(17,3+(i-1)*6);
172  A(urz_bdy{i}(2),pphiz_bdy{3}(2)) = A(urz_bdy{i}(2),pphiz_bdy{3}(2)) ...
      - Lval(18,3+(i-1)*6);
173
174  A(uphix_bdy{i}(2),prx_bdy{1}(2)) = A(uphix_bdy{i}(2),prx_bdy{1}(2)) ...
      - Lval(1,4+(i-1)*6);
175  A(uphix_bdy{i}(2),pry_bdy{1}(2)) = A(uphix_bdy{i}(2),pry_bdy{1}(2)) ...
      - Lval(2,4+(i-1)*6);
176  A(uphix_bdy{i}(2),prz_bdy{1}(2)) = A(uphix_bdy{i}(2),prz_bdy{1}(2)) ...
      - Lval(3,4+(i-1)*6);
177  A(uphix_bdy{i}(2),pphix_bdy{1}(2)) = ...
      A(uphix_bdy{i}(2),pphix_bdy{1}(2)) - Lval(4,4+(i-1)*6);
178  A(uphix_bdy{i}(2),pphiy_bdy{1}(2)) = ...
      A(uphix_bdy{i}(2),pphiy_bdy{1}(2)) - Lval(5,4+(i-1)*6);
179  A(uphix_bdy{i}(2),pphiz_bdy{1}(2)) = ...
      A(uphix_bdy{i}(2),pphiz_bdy{1}(2)) - Lval(6,4+(i-1)*6);
180  A(uphix_bdy{i}(2),prx_bdy{2}(2)) = A(uphix_bdy{i}(2),prx_bdy{2}(2)) ...
      - Lval(7,4+(i-1)*6);
181  A(uphix_bdy{i}(2),pry_bdy{2}(2)) = A(uphix_bdy{i}(2),pry_bdy{2}(2)) ...
      - Lval(8,4+(i-1)*6);
182  A(uphix_bdy{i}(2),prz_bdy{2}(2)) = A(uphix_bdy{i}(2),prz_bdy{2}(2)) ...
      - Lval(9,4+(i-1)*6);
183  A(uphix_bdy{i}(2),pphix_bdy{2}(2)) = ...
      A(uphix_bdy{i}(2),pphix_bdy{2}(2)) - Lval(10,4+(i-1)*6);
184  A(uphix_bdy{i}(2),pphiy_bdy{2}(2)) = ...
      A(uphix_bdy{i}(2),pphiy_bdy{2}(2)) - Lval(11,4+(i-1)*6);
185  A(uphix_bdy{i}(2),pphiz_bdy{2}(2)) = ...
      A(uphix_bdy{i}(2),pphiz_bdy{2}(2)) - Lval(12,4+(i-1)*6);
186  A(uphix_bdy{i}(2),prx_bdy{3}(2)) = A(uphix_bdy{i}(2),prx_bdy{3}(2)) ...
      - Lval(13,4+(i-1)*6);
187  A(uphix_bdy{i}(2),pry_bdy{3}(2)) = A(uphix_bdy{i}(2),pry_bdy{3}(2)) ...
      - Lval(14,4+(i-1)*6);
188  A(uphix_bdy{i}(2),prz_bdy{3}(2)) = A(uphix_bdy{i}(2),prz_bdy{3}(2)) ...
      - Lval(15,4+(i-1)*6);
189  A(uphix_bdy{i}(2),pphix_bdy{3}(2)) = ...
      A(uphix_bdy{i}(2),pphix_bdy{3}(2)) - Lval(16,4+(i-1)*6);
190  A(uphix_bdy{i}(2),pphiy_bdy{3}(2)) = ...
      A(uphix_bdy{i}(2),pphiy_bdy{3}(2)) - Lval(17,4+(i-1)*6);
191  A(uphix_bdy{i}(2),pphiz_bdy{3}(2)) = ...
      A(uphix_bdy{i}(2),pphiz_bdy{3}(2)) - Lval(18,4+(i-1)*6);
192
193  A(uphiy_bdy{i}(2),prx_bdy{1}(2)) = A(uphiy_bdy{i}(2),prx_bdy{1}(2)) ...
      - Lval(1,5+(i-1)*6);
194  A(uphiy_bdy{i}(2),pry_bdy{1}(2)) = A(uphiy_bdy{i}(2),pry_bdy{1}(2)) ...
      - Lval(2,5+(i-1)*6);
195  A(uphiy_bdy{i}(2),prz_bdy{1}(2)) = A(uphiy_bdy{i}(2),prz_bdy{1}(2)) ...
      - Lval(3,5+(i-1)*6);
196  A(uphiy_bdy{i}(2),pphix_bdy{1}(2)) = ...
      A(uphiy_bdy{i}(2),pphix_bdy{1}(2)) - Lval(4,5+(i-1)*6);
197  A(uphiy_bdy{i}(2),pphiy_bdy{1}(2)) = ...

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    A(uphiy_bdy{i}(2),pphiy_bdy{1}(2)) - Lval(5,5+(i-1)*6);
198 A(uphiy_bdy{i}(2),pphiz_bdy{1}(2)) = ...
    A(uphiy_bdy{i}(2),pphiz_bdy{1}(2)) - Lval(6,5+(i-1)*6);
199 A(uphiy_bdy{i}(2),prx_bdy{2}(2)) = A(uphiy_bdy{i}(2),prx_bdy{2}(2)) ...
    - Lval(7,5+(i-1)*6);
200 A(uphiy_bdy{i}(2),pry_bdy{2}(2)) = A(uphiy_bdy{i}(2),pry_bdy{2}(2)) ...
    - Lval(8,5+(i-1)*6);
201 A(uphiy_bdy{i}(2),prz_bdy{2}(2)) = A(uphiy_bdy{i}(2),prz_bdy{2}(2)) ...
    - Lval(9,5+(i-1)*6);
202 A(uphiy_bdy{i}(2),pphix_bdy{2}(2)) = ...
    A(uphiy_bdy{i}(2),pphix_bdy{2}(2)) - Lval(10,5+(i-1)*6);
203 A(uphiy_bdy{i}(2),pphiy_bdy{2}(2)) = ...
    A(uphiy_bdy{i}(2),pphiy_bdy{2}(2)) - Lval(11,5+(i-1)*6);
204 A(uphiy_bdy{i}(2),pphiz_bdy{2}(2)) = ...
    A(uphiy_bdy{i}(2),pphiz_bdy{2}(2)) - Lval(12,5+(i-1)*6);
205 A(uphiy_bdy{i}(2),prx_bdy{3}(2)) = A(uphiy_bdy{i}(2),prx_bdy{3}(2)) ...
    - Lval(13,5+(i-1)*6);
206 A(uphiy_bdy{i}(2),pry_bdy{3}(2)) = A(uphiy_bdy{i}(2),pry_bdy{3}(2)) ...
    - Lval(14,5+(i-1)*6);
207 A(uphiy_bdy{i}(2),prz_bdy{3}(2)) = A(uphiy_bdy{i}(2),prz_bdy{3}(2)) ...
    - Lval(15,5+(i-1)*6);
208 A(uphiy_bdy{i}(2),pphix_bdy{3}(2)) = ...
    A(uphiy_bdy{i}(2),pphix_bdy{3}(2)) - Lval(16,5+(i-1)*6);
209 A(uphiy_bdy{i}(2),pphiy_bdy{3}(2)) = ...
    A(uphiy_bdy{i}(2),pphiy_bdy{3}(2)) - Lval(17,5+(i-1)*6);
210 A(uphiy_bdy{i}(2),pphiz_bdy{3}(2)) = ...
    A(uphiy_bdy{i}(2),pphiz_bdy{3}(2)) - Lval(18,5+(i-1)*6);
211
212 A(uphiz_bdy{i}(2),prx_bdy{1}(2)) = A(uphiz_bdy{i}(2),prx_bdy{1}(2)) ...
    - Lval(1,6+(i-1)*6);
213 A(uphiz_bdy{i}(2),pry_bdy{1}(2)) = A(uphiz_bdy{i}(2),pry_bdy{1}(2)) ...
    - Lval(2,6+(i-1)*6);
214 A(uphiz_bdy{i}(2),prz_bdy{1}(2)) = A(uphiz_bdy{i}(2),prz_bdy{1}(2)) ...
    - Lval(3,6+(i-1)*6);
215 A(uphiz_bdy{i}(2),pphix_bdy{1}(2)) = ...
    A(uphiz_bdy{i}(2),pphix_bdy{1}(2)) - Lval(4,6+(i-1)*6);
216 A(uphiz_bdy{i}(2),pphiy_bdy{1}(2)) = ...
    A(uphiz_bdy{i}(2),pphiy_bdy{1}(2)) - Lval(5,6+(i-1)*6);
217 A(uphiz_bdy{i}(2),pphiz_bdy{1}(2)) = ...
    A(uphiz_bdy{i}(2),pphiz_bdy{1}(2)) - Lval(6,6+(i-1)*6);
218 A(uphiz_bdy{i}(2),prx_bdy{2}(2)) = A(uphiz_bdy{i}(2),prx_bdy{2}(2)) ...
    - Lval(7,6+(i-1)*6);
219 A(uphiz_bdy{i}(2),pry_bdy{2}(2)) = A(uphiz_bdy{i}(2),pry_bdy{2}(2)) ...
    - Lval(8,6+(i-1)*6);
220 A(uphiz_bdy{i}(2),prz_bdy{2}(2)) = A(uphiz_bdy{i}(2),prz_bdy{2}(2)) ...
    - Lval(9,6+(i-1)*6);
221 A(uphiz_bdy{i}(2),pphix_bdy{2}(2)) = ...
    A(uphiz_bdy{i}(2),pphix_bdy{2}(2)) - Lval(10,6+(i-1)*6);
222 A(uphiz_bdy{i}(2),pphiy_bdy{2}(2)) = ...
    A(uphiz_bdy{i}(2),pphiy_bdy{2}(2)) - Lval(11,6+(i-1)*6);
223 A(uphiz_bdy{i}(2),pphiz_bdy{2}(2)) = ...

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    A(uphiz_bdy{i}(2),pphiz_bdy{2}(2)) - Lval(12,6+(i-1)*6);
224 A(uphiz_bdy{i}(2),prx_bdy{3}(2)) = A(uphiz_bdy{i}(2),prx_bdy{3}(2)) ...
    - Lval(13,6+(i-1)*6);
225 A(uphiz_bdy{i}(2),pry_bdy{3}(2)) = A(uphiz_bdy{i}(2),pry_bdy{3}(2)) ...
    - Lval(14,6+(i-1)*6);
226 A(uphiz_bdy{i}(2),prz_bdy{3}(2)) = A(uphiz_bdy{i}(2),prz_bdy{3}(2)) ...
    - Lval(15,6+(i-1)*6);
227 A(uphiz_bdy{i}(2),pphix_bdy{3}(2)) = ...
    A(uphiz_bdy{i}(2),pphix_bdy{3}(2)) - Lval(16,6+(i-1)*6);
228 A(uphiz_bdy{i}(2),pphiy_bdy{3}(2)) = ...
    A(uphiz_bdy{i}(2),pphiy_bdy{3}(2)) - Lval(17,6+(i-1)*6);
229 A(uphiz_bdy{i}(2),pphiz_bdy{3}(2)) = ...
    A(uphiz_bdy{i}(2),pphiz_bdy{3}(2)) - Lval(18,6+(i-1)*6);
230 end
231
232 for i = 1 : 3
233     A(prx_bdy{i}(2),urx_bdy{1}(2)) = A(prx_bdy{i}(2),urx_bdy{1}(2)) - ...
        Lval(1,1+(i-1)*6);
234     A(prx_bdy{i}(2),ury_bdy{1}(2)) = A(prx_bdy{i}(2),ury_bdy{1}(2)) - ...
        Lval(2,1+(i-1)*6);
235     A(prx_bdy{i}(2),urz_bdy{1}(2)) = A(prx_bdy{i}(2),urz_bdy{1}(2)) - ...
        Lval(3,1+(i-1)*6);
236     A(prx_bdy{i}(2),uphix_bdy{1}(2)) = A(prx_bdy{i}(2),uphix_bdy{1}(2)) ...
        - Lval(4,1+(i-1)*6);
237     A(prx_bdy{i}(2),uphiy_bdy{1}(2)) = A(prx_bdy{i}(2),uphiy_bdy{1}(2)) ...
        - Lval(5,1+(i-1)*6);
238     A(prx_bdy{i}(2),uphiz_bdy{1}(2)) = A(prx_bdy{i}(2),uphiz_bdy{1}(2)) ...
        - Lval(6,1+(i-1)*6);
239     A(prx_bdy{i}(2),urx_bdy{2}(2)) = A(prx_bdy{i}(2),urx_bdy{2}(2)) - ...
        Lval(7,1+(i-1)*6);
240     A(prx_bdy{i}(2),ury_bdy{2}(2)) = A(prx_bdy{i}(2),ury_bdy{2}(2)) - ...
        Lval(8,1+(i-1)*6);
241     A(prx_bdy{i}(2),urz_bdy{2}(2)) = A(prx_bdy{i}(2),urz_bdy{2}(2)) - ...
        Lval(9,1+(i-1)*6);
242     A(prx_bdy{i}(2),uphix_bdy{2}(2)) = A(prx_bdy{i}(2),uphix_bdy{2}(2)) ...
        - Lval(10,1+(i-1)*6);
243     A(prx_bdy{i}(2),uphiy_bdy{2}(2)) = A(prx_bdy{i}(2),uphiy_bdy{2}(2)) ...
        - Lval(11,1+(i-1)*6);
244     A(prx_bdy{i}(2),uphiz_bdy{2}(2)) = A(prx_bdy{i}(2),uphiz_bdy{2}(2)) ...
        - Lval(12,1+(i-1)*6);
245     A(prx_bdy{i}(2),urx_bdy{3}(2)) = A(prx_bdy{i}(2),urx_bdy{3}(2)) - ...
        Lval(13,1+(i-1)*6);
246     A(prx_bdy{i}(2),ury_bdy{3}(2)) = A(prx_bdy{i}(2),ury_bdy{3}(2)) - ...
        Lval(14,1+(i-1)*6);
247     A(prx_bdy{i}(2),urz_bdy{3}(2)) = A(prx_bdy{i}(2),urz_bdy{3}(2)) - ...
        Lval(15,1+(i-1)*6);
248     A(prx_bdy{i}(2),uphix_bdy{3}(2)) = A(prx_bdy{i}(2),uphix_bdy{3}(2)) ...
        - Lval(16,1+(i-1)*6);
249     A(prx_bdy{i}(2),uphiy_bdy{3}(2)) = A(prx_bdy{i}(2),uphiy_bdy{3}(2)) ...
        - Lval(17,1+(i-1)*6);
250     A(prx_bdy{i}(2),uphiz_bdy{3}(2)) = A(prx_bdy{i}(2),uphiz_bdy{3}(2)) ...

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- Lval(18,1+(i-1)*6);
251
252 A(pry_bdy{i}(2),urx_bdy{1}(2)) = A(pry_bdy{i}(2),urx_bdy{1}(2)) - ...
    Lval(1,2+(i-1)*6);
253 A(pry_bdy{i}(2),ury_bdy{1}(2)) = A(pry_bdy{i}(2),ury_bdy{1}(2)) - ...
    Lval(2,2+(i-1)*6);
254 A(pry_bdy{i}(2),urz_bdy{1}(2)) = A(pry_bdy{i}(2),urz_bdy{1}(2)) - ...
    Lval(3,2+(i-1)*6);
255 A(pry_bdy{i}(2),uphix_bdy{1}(2)) = A(pry_bdy{i}(2),uphix_bdy{1}(2)) ...
    - Lval(4,2+(i-1)*6);
256 A(pry_bdy{i}(2),uphiy_bdy{1}(2)) = A(pry_bdy{i}(2),uphiy_bdy{1}(2)) ...
    - Lval(5,2+(i-1)*6);
257 A(pry_bdy{i}(2),uphiz_bdy{1}(2)) = A(pry_bdy{i}(2),uphiz_bdy{1}(2)) ...
    - Lval(6,2+(i-1)*6);
258 A(pry_bdy{i}(2),urx_bdy{2}(2)) = A(pry_bdy{i}(2),urx_bdy{2}(2)) - ...
    Lval(7,2+(i-1)*6);
259 A(pry_bdy{i}(2),ury_bdy{2}(2)) = A(pry_bdy{i}(2),ury_bdy{2}(2)) - ...
    Lval(8,2+(i-1)*6);
260 A(pry_bdy{i}(2),urz_bdy{2}(2)) = A(pry_bdy{i}(2),urz_bdy{2}(2)) - ...
    Lval(9,2+(i-1)*6);
261 A(pry_bdy{i}(2),uphix_bdy{2}(2)) = A(pry_bdy{i}(2),uphix_bdy{2}(2)) ...
    - Lval(10,2+(i-1)*6);
262 A(pry_bdy{i}(2),uphiy_bdy{2}(2)) = A(pry_bdy{i}(2),uphiy_bdy{2}(2)) ...
    - Lval(11,2+(i-1)*6);
263 A(pry_bdy{i}(2),uphiz_bdy{2}(2)) = A(pry_bdy{i}(2),uphiz_bdy{2}(2)) ...
    - Lval(12,2+(i-1)*6);
264 A(pry_bdy{i}(2),urx_bdy{3}(2)) = A(pry_bdy{i}(2),urx_bdy{3}(2)) - ...
    Lval(13,2+(i-1)*6);
265 A(pry_bdy{i}(2),ury_bdy{3}(2)) = A(pry_bdy{i}(2),ury_bdy{3}(2)) - ...
    Lval(14,2+(i-1)*6);
266 A(pry_bdy{i}(2),urz_bdy{3}(2)) = A(pry_bdy{i}(2),urz_bdy{3}(2)) - ...
    Lval(15,2+(i-1)*6);
267 A(pry_bdy{i}(2),uphix_bdy{3}(2)) = A(pry_bdy{i}(2),uphix_bdy{3}(2)) ...
    - Lval(16,2+(i-1)*6);
268 A(pry_bdy{i}(2),uphiy_bdy{3}(2)) = A(pry_bdy{i}(2),uphiy_bdy{3}(2)) ...
    - Lval(17,2+(i-1)*6);
269 A(pry_bdy{i}(2),uphiz_bdy{3}(2)) = A(pry_bdy{i}(2),uphiz_bdy{3}(2)) ...
    - Lval(18,2+(i-1)*6);
270
271 A(prz_bdy{i}(2),urx_bdy{1}(2)) = A(prz_bdy{i}(2),urx_bdy{1}(2)) - ...
    Lval(1,3+(i-1)*6);
272 A(prz_bdy{i}(2),ury_bdy{1}(2)) = A(prz_bdy{i}(2),ury_bdy{1}(2)) - ...
    Lval(2,3+(i-1)*6);
273 A(prz_bdy{i}(2),urz_bdy{1}(2)) = A(prz_bdy{i}(2),urz_bdy{1}(2)) - ...
    Lval(3,3+(i-1)*6);
274 A(prz_bdy{i}(2),uphix_bdy{1}(2)) = A(prz_bdy{i}(2),uphix_bdy{1}(2)) ...
    - Lval(4,3+(i-1)*6);
275 A(prz_bdy{i}(2),uphiy_bdy{1}(2)) = A(prz_bdy{i}(2),uphiy_bdy{1}(2)) ...
    - Lval(5,3+(i-1)*6);
276 A(prz_bdy{i}(2),uphiz_bdy{1}(2)) = A(prz_bdy{i}(2),uphiz_bdy{1}(2)) ...
    - Lval(6,3+(i-1)*6);

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277 A(prz_bdy{i}(2), urx_bdy{2}(2)) = A(prz_bdy{i}(2), urx_bdy{2}(2)) - ...
    Lval(7, 3+(i-1)*6);
278 A(prz_bdy{i}(2), ury_bdy{2}(2)) = A(prz_bdy{i}(2), ury_bdy{2}(2)) - ...
    Lval(8, 3+(i-1)*6);
279 A(prz_bdy{i}(2), urz_bdy{2}(2)) = A(prz_bdy{i}(2), urz_bdy{2}(2)) - ...
    Lval(9, 3+(i-1)*6);
280 A(prz_bdy{i}(2), uphix_bdy{2}(2)) = A(prz_bdy{i}(2), uphix_bdy{2}(2)) ...
    - Lval(10, 3+(i-1)*6);
281 A(prz_bdy{i}(2), uphiy_bdy{2}(2)) = A(prz_bdy{i}(2), uphiy_bdy{2}(2)) ...
    - Lval(11, 3+(i-1)*6);
282 A(prz_bdy{i}(2), uphiz_bdy{2}(2)) = A(prz_bdy{i}(2), uphiz_bdy{2}(2)) ...
    - Lval(12, 3+(i-1)*6);
283 A(prz_bdy{i}(2), urx_bdy{3}(2)) = A(prz_bdy{i}(2), urx_bdy{3}(2)) - ...
    Lval(13, 3+(i-1)*6);
284 A(prz_bdy{i}(2), ury_bdy{3}(2)) = A(prz_bdy{i}(2), ury_bdy{3}(2)) - ...
    Lval(14, 3+(i-1)*6);
285 A(prz_bdy{i}(2), urz_bdy{3}(2)) = A(prz_bdy{i}(2), urz_bdy{3}(2)) - ...
    Lval(15, 3+(i-1)*6);
286 A(prz_bdy{i}(2), uphix_bdy{3}(2)) = A(prz_bdy{i}(2), uphix_bdy{3}(2)) ...
    - Lval(16, 3+(i-1)*6);
287 A(prz_bdy{i}(2), uphiy_bdy{3}(2)) = A(prz_bdy{i}(2), uphiy_bdy{3}(2)) ...
    - Lval(17, 3+(i-1)*6);
288 A(prz_bdy{i}(2), uphiz_bdy{3}(2)) = A(prz_bdy{i}(2), uphiz_bdy{3}(2)) ...
    - Lval(18, 3+(i-1)*6);
289
290 A(pphix_bdy{i}(2), urx_bdy{1}(2)) = A(pphix_bdy{i}(2), urx_bdy{1}(2)) ...
    - Lval(1, 4+(i-1)*6);
291 A(pphix_bdy{i}(2), ury_bdy{1}(2)) = A(pphix_bdy{i}(2), ury_bdy{1}(2)) ...
    - Lval(2, 4+(i-1)*6);
292 A(pphix_bdy{i}(2), urz_bdy{1}(2)) = A(pphix_bdy{i}(2), urz_bdy{1}(2)) ...
    - Lval(3, 4+(i-1)*6);
293 A(pphix_bdy{i}(2), uphix_bdy{1}(2)) = ...
    A(pphix_bdy{i}(2), uphix_bdy{1}(2)) - Lval(4, 4+(i-1)*6);
294 A(pphix_bdy{i}(2), uphiy_bdy{1}(2)) = ...
    A(pphix_bdy{i}(2), uphiy_bdy{1}(2)) - Lval(5, 4+(i-1)*6);
295 A(pphix_bdy{i}(2), uphiz_bdy{1}(2)) = ...
    A(pphix_bdy{i}(2), uphiz_bdy{1}(2)) - Lval(6, 4+(i-1)*6);
296 A(pphix_bdy{i}(2), urx_bdy{2}(2)) = A(pphix_bdy{i}(2), urx_bdy{2}(2)) ...
    - Lval(7, 4+(i-1)*6);
297 A(pphix_bdy{i}(2), ury_bdy{2}(2)) = A(pphix_bdy{i}(2), ury_bdy{2}(2)) ...
    - Lval(8, 4+(i-1)*6);
298 A(pphix_bdy{i}(2), urz_bdy{2}(2)) = A(pphix_bdy{i}(2), urz_bdy{2}(2)) ...
    - Lval(9, 4+(i-1)*6);
299 A(pphix_bdy{i}(2), uphix_bdy{2}(2)) = ...
    A(pphix_bdy{i}(2), uphix_bdy{2}(2)) - Lval(10, 4+(i-1)*6);
300 A(pphix_bdy{i}(2), uphiy_bdy{2}(2)) = ...
    A(pphix_bdy{i}(2), uphiy_bdy{2}(2)) - Lval(11, 4+(i-1)*6);
301 A(pphix_bdy{i}(2), uphiz_bdy{2}(2)) = ...
    A(pphix_bdy{i}(2), uphiz_bdy{2}(2)) - Lval(12, 4+(i-1)*6);
302 A(pphix_bdy{i}(2), urx_bdy{3}(2)) = A(pphix_bdy{i}(2), urx_bdy{3}(2)) ...
    - Lval(13, 4+(i-1)*6);

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303  A(pphix_bdy{i}(2), ury_bdy{3}(2)) = A(pphix_bdy{i}(2), ury_bdy{3}(2)) ...
      - Lval(14, 4+(i-1)*6);
304  A(pphix_bdy{i}(2), urz_bdy{3}(2)) = A(pphix_bdy{i}(2), urz_bdy{3}(2)) ...
      - Lval(15, 4+(i-1)*6);
305  A(pphix_bdy{i}(2), uphix_bdy{3}(2)) = ...
      A(pphix_bdy{i}(2), uphix_bdy{3}(2)) - Lval(16, 4+(i-1)*6);
306  A(pphix_bdy{i}(2), uphiy_bdy{3}(2)) = ...
      A(pphix_bdy{i}(2), uphiy_bdy{3}(2)) - Lval(17, 4+(i-1)*6);
307  A(pphix_bdy{i}(2), uphiz_bdy{3}(2)) = ...
      A(pphix_bdy{i}(2), uphiz_bdy{3}(2)) - Lval(18, 4+(i-1)*6);
308
309  A(pphiy_bdy{i}(2), urx_bdy{1}(2)) = A(pphiy_bdy{i}(2), urx_bdy{1}(2)) ...
      - Lval(1, 5+(i-1)*6);
310  A(pphiy_bdy{i}(2), ury_bdy{1}(2)) = A(pphiy_bdy{i}(2), ury_bdy{1}(2)) ...
      - Lval(2, 5+(i-1)*6);
311  A(pphiy_bdy{i}(2), urz_bdy{1}(2)) = A(pphiy_bdy{i}(2), urz_bdy{1}(2)) ...
      - Lval(3, 5+(i-1)*6);
312  A(pphiy_bdy{i}(2), uphix_bdy{1}(2)) = ...
      A(pphiy_bdy{i}(2), uphix_bdy{1}(2)) - Lval(4, 5+(i-1)*6);
313  A(pphiy_bdy{i}(2), uphiy_bdy{1}(2)) = ...
      A(pphiy_bdy{i}(2), uphiy_bdy{1}(2)) - Lval(5, 5+(i-1)*6);
314  A(pphiy_bdy{i}(2), uphiz_bdy{1}(2)) = ...
      A(pphiy_bdy{i}(2), uphiz_bdy{1}(2)) - Lval(6, 5+(i-1)*6);
315  A(pphiy_bdy{i}(2), urx_bdy{2}(2)) = A(pphiy_bdy{i}(2), urx_bdy{2}(2)) ...
      - Lval(7, 5+(i-1)*6);
316  A(pphiy_bdy{i}(2), ury_bdy{2}(2)) = A(pphiy_bdy{i}(2), ury_bdy{2}(2)) ...
      - Lval(8, 5+(i-1)*6);
317  A(pphiy_bdy{i}(2), urz_bdy{2}(2)) = A(pphiy_bdy{i}(2), urz_bdy{2}(2)) ...
      - Lval(9, 5+(i-1)*6);
318  A(pphiy_bdy{i}(2), uphix_bdy{2}(2)) = ...
      A(pphiy_bdy{i}(2), uphix_bdy{2}(2)) - Lval(10, 5+(i-1)*6);
319  A(pphiy_bdy{i}(2), uphiy_bdy{2}(2)) = ...
      A(pphiy_bdy{i}(2), uphiy_bdy{2}(2)) - Lval(11, 5+(i-1)*6);
320  A(pphiy_bdy{i}(2), uphiz_bdy{2}(2)) = ...
      A(pphiy_bdy{i}(2), uphiz_bdy{2}(2)) - Lval(12, 5+(i-1)*6);
321  A(pphiy_bdy{i}(2), urx_bdy{3}(2)) = A(pphiy_bdy{i}(2), urx_bdy{3}(2)) ...
      - Lval(13, 5+(i-1)*6);
322  A(pphiy_bdy{i}(2), ury_bdy{3}(2)) = A(pphiy_bdy{i}(2), ury_bdy{3}(2)) ...
      - Lval(14, 5+(i-1)*6);
323  A(pphiy_bdy{i}(2), urz_bdy{3}(2)) = A(pphiy_bdy{i}(2), urz_bdy{3}(2)) ...
      - Lval(15, 5+(i-1)*6);
324  A(pphiy_bdy{i}(2), uphix_bdy{3}(2)) = ...
      A(pphiy_bdy{i}(2), uphix_bdy{3}(2)) - Lval(16, 5+(i-1)*6);
325  A(pphiy_bdy{i}(2), uphiy_bdy{3}(2)) = ...
      A(pphiy_bdy{i}(2), uphiy_bdy{3}(2)) - Lval(17, 5+(i-1)*6);
326  A(pphiy_bdy{i}(2), uphiz_bdy{3}(2)) = ...
      A(pphiy_bdy{i}(2), uphiz_bdy{3}(2)) - Lval(18, 5+(i-1)*6);
327
328  A(pphiz_bdy{i}(2), urx_bdy{1}(2)) = A(pphiz_bdy{i}(2), urx_bdy{1}(2)) ...
      - Lval(1, 6+(i-1)*6);
329  A(pphiz_bdy{i}(2), ury_bdy{1}(2)) = A(pphiz_bdy{i}(2), ury_bdy{1}(2)) ...

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- Lval(2, 6+(i-1)*6);
330 A(pphiz_bdy{i}(2), urz_bdy{1}(2)) = A(pphiz_bdy{i}(2), urz_bdy{1}(2)) ...
- Lval(3, 6+(i-1)*6);
331 A(pphiz_bdy{i}(2), uphix_bdy{1}(2)) = ...
A(pphiz_bdy{i}(2), uphix_bdy{1}(2)) - Lval(4, 6+(i-1)*6);
332 A(pphiz_bdy{i}(2), uphiy_bdy{1}(2)) = ...
A(pphiz_bdy{i}(2), uphiy_bdy{1}(2)) - Lval(5, 6+(i-1)*6);
333 A(pphiz_bdy{i}(2), uphiz_bdy{1}(2)) = ...
A(pphiz_bdy{i}(2), uphiz_bdy{1}(2)) - Lval(6, 6+(i-1)*6);
334 A(pphiz_bdy{i}(2), urx_bdy{2}(2)) = A(pphiz_bdy{i}(2), urx_bdy{2}(2)) ...
- Lval(7, 6+(i-1)*6);
335 A(pphiz_bdy{i}(2), ury_bdy{2}(2)) = A(pphiz_bdy{i}(2), ury_bdy{2}(2)) ...
- Lval(8, 6+(i-1)*6);
336 A(pphiz_bdy{i}(2), urz_bdy{2}(2)) = A(pphiz_bdy{i}(2), urz_bdy{2}(2)) ...
- Lval(9, 6+(i-1)*6);
337 A(pphiz_bdy{i}(2), uphix_bdy{2}(2)) = ...
A(pphiz_bdy{i}(2), uphix_bdy{2}(2)) - Lval(10, 6+(i-1)*6);
338 A(pphiz_bdy{i}(2), uphiy_bdy{2}(2)) = ...
A(pphiz_bdy{i}(2), uphiy_bdy{2}(2)) - Lval(11, 6+(i-1)*6);
339 A(pphiz_bdy{i}(2), uphiz_bdy{2}(2)) = ...
A(pphiz_bdy{i}(2), uphiz_bdy{2}(2)) - Lval(12, 6+(i-1)*6);
340 A(pphiz_bdy{i}(2), urx_bdy{3}(2)) = A(pphiz_bdy{i}(2), urx_bdy{3}(2)) ...
- Lval(13, 6+(i-1)*6);
341 A(pphiz_bdy{i}(2), ury_bdy{3}(2)) = A(pphiz_bdy{i}(2), ury_bdy{3}(2)) ...
- Lval(14, 6+(i-1)*6);
342 A(pphiz_bdy{i}(2), urz_bdy{3}(2)) = A(pphiz_bdy{i}(2), urz_bdy{3}(2)) ...
- Lval(15, 6+(i-1)*6);
343 A(pphiz_bdy{i}(2), uphix_bdy{3}(2)) = ...
A(pphiz_bdy{i}(2), uphix_bdy{3}(2)) - Lval(16, 6+(i-1)*6);
344 A(pphiz_bdy{i}(2), uphiy_bdy{3}(2)) = ...
A(pphiz_bdy{i}(2), uphiy_bdy{3}(2)) - Lval(17, 6+(i-1)*6);
345 A(pphiz_bdy{i}(2), uphiz_bdy{3}(2)) = ...
A(pphiz_bdy{i}(2), uphiz_bdy{3}(2)) - Lval(18, 6+(i-1)*6);
346 end
347
348 syms x
349 for i = 1 : 3
350 z{i} = ulocal{i};
351 zNum{i} = double(subs(z{i}, x01));
352 dz{i} = diff(z{i}, x);
353 dzNum{i} = double(subs(dz{i}, x01));
354 dzNum{i}(:, end) = double(subs(dz{i}, x, 0.999999));
355 dzNum{i}(:, 1) = double(subs(dz{i}, x, 0.000001));
356
357 fx(i) = q{i}(1);
358 dfx{i} = diff(fx(i), x);
359 dfxNum{i} = double(subs(dfx{i}, x01));
360 dfxNum{i}(:, end) = double(subs(dfx{i}, x, 0.999999));
361 dfxNum{i}(:, 1) = double(subs(dfx{i}, x, 0.000001));
362
363 fy(i) = q{i}(2);

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364     dfy{i} = diff(fy(i),x);
365     dfyNum{i} = double(subs(dfy{i},x01));
366     dfyNum{i}(:,end) = double(subs(dfy{i}, x, 0.999999));
367     dfyNum{i}(:,1) = double(subs(dfy{i}, x, 0.000001));
368
369     fz(i) = q{i}(3);
370     dfz{i} = diff(fz(i),x);
371     dfzNum{i} = double(subs(dfz{i},x01));
372     dfzNum{i}(:,end) = double(subs(dfz{i}, x, 0.999999));
373     dfzNum{i}(:,1) = double(subs(dfz{i}, x, 0.000001));
374
375     mx(i) = q{i}(4);
376     dm{x}{i} = diff(mx(i),x);
377     dm{x}{i} = double(subs(dm{x}{i},x01));
378     dm{x}{i}(:,end) = double(subs(dm{x}{i}, x, 0.999999));
379     dm{x}{i}(:,1) = double(subs(dm{x}{i}, x, 0.000001));
380
381     my(i) = q{i}(5);
382     dmy{i} = diff(my(i),x);
383     dmyNum{i} = double(subs(dmy{i},x01));
384     dmyNum{i}(:,end) = double(subs(dmy{i}, x, 0.999999));
385     dmyNum{i}(:,1) = double(subs(dmy{i}, x, 0.000001));
386
387     mz(i) = q{i}(6);
388     dmz{i} = diff(mz(i),x);
389     dmzNum{i} = double(subs(dmz{i},x01));
390     dmzNum{i}(:,end) = double(subs(dmz{i}, x, 0.999999));
391     dmzNum{i}(:,1) = double(subs(dmz{i}, x, 0.000001));
392 end
393
394 bold = zeros(size(A,1),1);
395
396 for i = 1 : 3
397     bold(urx_bdy{i}(1):urx_bdy{i}(4)) = A4*[zNum{i}(1,:) ...
398         dzNum{i}(1,:)*hmax]';
399     bold(ury_bdy{i}(1):ury_bdy{i}(4)) = A4*[zNum{i}(2,:) ...
400         dzNum{i}(2,:)*hmax]';
401     bold(urz_bdy{i}(1):urz_bdy{i}(4)) = A4*[zNum{i}(3,:) ...
402         dzNum{i}(3,:)*hmax]';
403     bold(uphix_bdy{i}(1):uphix_bdy{i}(4)) = A4*[zNum{i}(4,:) ...
404         dzNum{i}(4,:)*hmax]';
405     bold(uphiy_bdy{i}(1):uphiy_bdy{i}(4)) = A4*[zNum{i}(5,:) ...
406         dzNum{i}(5,:)*hmax]';
407     bold(uphiz_bdy{i}(1):uphiz_bdy{i}(4)) = A4*[zNum{i}(6,:) ...
408         dzNum{i}(6,:)*hmax]';
409
410     bold(prx_bdy{i}(1):prx_bdy{i}(4)) = A4*[double(subs(fx(i),x01)) ...
411         dfxNum{i}*hmax]';
412     bold(pry_bdy{i}(1):pry_bdy{i}(4)) = A4*[double(subs(fy(i),x01)) ...
413         dfyNum{i}*hmax]';
414     bold(prz_bdy{i}(1):prz_bdy{i}(4)) = A4*[double(subs(fz(i),x01)) ...

```

```

    dfzNum{i}*hmax]';
407 bold(pphix_bdy{i}(1):pphix_bdy{i}(4)) = A4*[double(subs(mx(i),x01)) ...
    dmxNum{i}*hmax]';
408 bold(pphiy_bdy{i}(1):pphiy_bdy{i}(4)) = A4*[double(subs(my(i),x01)) ...
    dmyNum{i}*hmax]';
409 bold(pphiz_bdy{i}(1):pphiz_bdy{i}(4)) = A4*[double(subs(mz(i),x01)) ...
    dmzNum{i}*hmax]';
410 end
411
412 b = bold;
413
414
415 % bounday condition
416 for i = 1 : 3
417     A(urx_bdy{i}(1),:) = 0;
418     A(urx_bdy{i}(1),urx_bdy{i}(1)) = 1;
419     b(urx_bdy{i}(1)) = 0;
420
421     A(ury_bdy{i}(1),:) = 0;
422     A(ury_bdy{i}(1),ury_bdy{i}(1)) = 1;
423     b(ury_bdy{i}(1)) = 0;
424
425     A(urz_bdy{i}(1),:) = 0;
426     A(urz_bdy{i}(1),urz_bdy{i}(1)) = 1;
427     b(urz_bdy{i}(1)) = u0(i);
428
429     A(uphix_bdy{i}(1),:) = 0;
430     A(uphix_bdy{i}(1),uphix_bdy{i}(1)) = 1;
431     b(uphix_bdy{i}(1)) = u0(i);
432
433     A(uphiy_bdy{i}(1),:) = 0;
434     A(uphiy_bdy{i}(1),uphiy_bdy{i}(1)) = 1;
435     b(uphiy_bdy{i}(1)) = u0(i);
436
437     A(uphiz_bdy{i}(1),:) = 0;
438     A(uphiz_bdy{i}(1),uphiz_bdy{i}(1)) = 1;
439     b(uphiz_bdy{i}(1)) = u0(i);
440
441     A(prx_bdy{i}(1),:) = 0;
442     A(prx_bdy{i}(1),prx_bdy{i}(1)) = 1;
443     b(prx_bdy{i}(1)) = 0;
444
445     A(pry_bdy{i}(1),:) = 0;
446     A(pry_bdy{i}(1),pry_bdy{i}(1)) = 1;
447     b(pry_bdy{i}(1)) = 0;
448
449     A(prz_bdy{i}(1),:) = 0;
450     A(prz_bdy{i}(1),prz_bdy{i}(1)) = 1;
451     b(prz_bdy{i}(1)) = u0(i);
452
453     A(pphix_bdy{i}(1),:) = 0;

```

```

454     A(pphix_bdy{i}(1),pphix_bdy{i}(1)) = 1;
455     b(pphix_bdy{i}(1)) = u0(i);
456
457     A(pphiy_bdy{i}(1), :) = 0;
458     A(pphiy_bdy{i}(1),pphiy_bdy{i}(1)) = 1;
459     b(pphiy_bdy{i}(1)) = u0(i);
460
461     A(pphiz_bdy{i}(1), :) = 0;
462     A(pphiz_bdy{i}(1),pphiz_bdy{i}(1)) = 1;
463     b(pphiz_bdy{i}(1)) = u0(i);
464
465     A(prz_bdy{i}(1), :) = 0;
466     A(prz_bdy{i}(1),prz_bdy{i}(1)) = 1;
467     b(prz_bdy{i}(1)) = u0(i);
468
469 end
470
471 solu = A\b;
472
473 for i = 1 : 3
474     urx{i} = solu(urx_bdy{i}(1):urx_bdy{i}(2));
475     ury{i} = solu(ury_bdy{i}(1):ury_bdy{i}(2));
476     urz{i} = solu(urz_bdy{i}(1):urz_bdy{i}(2));
477     uphix{i} = solu(uphix_bdy{i}(1):uphix_bdy{i}(2));
478     uphiy{i} = solu(uphiy_bdy{i}(1):uphiy_bdy{i}(2));
479     uphiz{i} = solu(uphiz_bdy{i}(1):uphiz_bdy{i}(2));
480
481     prx{i} = solu(prx_bdy{i}(1):prx_bdy{i}(2));
482     pry{i} = solu(pry_bdy{i}(1):pry_bdy{i}(2));
483     prz{i} = solu(prz_bdy{i}(1):prz_bdy{i}(2));
484     pphix{i} = solu(pphix_bdy{i}(1):pphix_bdy{i}(2));
485     pphiy{i} = solu(pphiy_bdy{i}(1):pphiy_bdy{i}(2));
486     pphiz{i} = solu(pphiz_bdy{i}(1):pphiz_bdy{i}(2));
487
488     vrx{i} = prx{i} + subs(fx(i),x01)';
489     vry{i} = pry{i} + subs(fy(i),x01)';
490     vrz{i} = prz{i} + subs(fz(i),x01)';
491     vphix{i} = pphix{i} + subs(mx(i),x01)';
492     vphiy{i} = pphiy{i} + subs(my(i),x01)';
493     vphiz{i} = pphiz{i} + subs(mz(i),x01)';
494 end
495
496 J = 0;
497
498 for i = 1 : 3
499     tmp = norm(urx{i}-double(subs(z{i}(1),x01))')^2*hmax;
500     J = J + tmp;
501     tmp = norm(ury{i}-double(subs(z{i}(2),x01))')^2*hmax;
502     J = J + tmp;
503     tmp = norm(urz{i}-double(subs(z{i}(3),x01))')^2*hmax;
504     J = J + tmp;

```

```

505     tmp = norm(uphix{i}-double(subs(z{i}(4),x01))')^2*hmax;
506     J = J + tmp;
507     tmp = norm(uphiy{i}-double(subs(z{i}(5),x01))')^2*hmax;
508     J = J + tmp;
509     tmp = norm(uphiz{i}-double(subs(z{i}(6),x01))')^2*hmax;
510     J = J + tmp;
511     tmp = norm(prx{i})^2*hmax+norm(pry{i})^2*hmax+norm(prz{i})^2*hmax;
512     J = J + tmp;
513     tmp = ...
           norm(pphix{i})^2*hmax+norm(pphiy{i})^2*hmax+norm(pphiz{i})^2*hmax;
514     J = J + tmp;
515 end
516
517 J = J/2;
518 end
519
520 function Eil = E(t)
521     Eil = [cos(t) sin(t) 0;
522           -sin(t) cos(t) 0;
523           0 0 1];
524 end

```

A.3 Wave Equation

In this section, we delve into the application of dynamic optimization techniques to the wave equation. Specifically, we explore example 1 in section 5.5.1, focusing on optimizing boundary control while verifying the Turnpike property.

```

1  % Example 1
2  % y0 = x;y1=0;
3
4  % Function to compute convergence using quadprog
5  % T: Time period
6  % J/T: Convergence ratio
7
8  function J = Example1(T)
9
10 % Set default value for T if not provided
11 if nargin == 0
12     T = 10;
13 end
14
15 % Parameter and time step
16 gamma = 0.1;
17 dt = 0.1;
18
19 % Number of time steps
20 M = T/dt;

```

```

21
22 % Number of expansion terms
23 N = 20;
24
25 % Gauss point and weights
26 s_ref = [-sqrt(3/5),0,sqrt(3/5)];
27 wt = [5,8,5]/9;
28
29 % Function handle for alpha0(x, n) = y0(x) * phin(x, n)
30 alpha0 = @(x, n) y0(x) .* phin(x,n);
31
32 % ////////// V sin //////////
33 VM = cell(M,1);
34 Vk = cell(M,1);
35
36 VMint = zeros(M,M);
37 Vkint = zeros(M,1);
38
39 tic;
40 for k = 1 : M
41     tk1 = dt*(k-1);
42     tk = dt*k;
43     VM{k} = @(t,n) ...
44         1/(pi/2+n*pi)*(cos((pi/2+n*pi)*(t-tk))-cos((pi/2+n*pi)*(t-tk1)));
45     Vk{k} = @(t,n) 1/(pi/2+n*pi)*(1-cos((pi/2+n*pi)*(t-tk1)));
46 end
47 elapsed_time = toc;
48 fprintf('Assembly of VMsin code block execution time: %.4f s\n', ...
49     elapsed_time);
50
51 % ////////// V cos //////////
52 VMc = cell(M,1);
53 Vkc = cell(M,1);
54
55 VMcint = zeros(M,M);
56 Vkcint = zeros(M,1);
57
58 tic;
59 for k = 1 : M
60     tk1 = dt*(k-1);
61     tk = dt*k;
62     VMc{k} = @(t,n) ...
63         1/(pi/2+n*pi)*(-sin((pi/2+n*pi)*(t-tk))+sin((pi/2+n*pi)*(t-tk1)));
64     Vkc{k} = @(t,n) 1/(pi/2+n*pi)*(sin((pi/2+n*pi)*(t-tk1)));
65 end
66 elapsed_time = toc;
67 fprintf('Assembly of VMcos code block execution time: %.4f s\n', ...
68     elapsed_time);
69
70 %% ===== q =====
71 % ///// q1 /////

```

```

68 tic;
69
70 for j = 1 : M
71     tj1 = dt*(j-1);
72     tj = dt*j;
73     t = (tj1+tj)/2;
74     for n = 0 : N
75         alpha0int = integral(@(x) alpha0(x, n), 0, 1);
76         for k = 1 : j-1
77             tmp = (-1)^n*2*sqrt(2)*alpha0int/(pi/2+n*pi)
78                 *cos((pi/2+n*pi)*(t)).*VM{k}(t,n)*dt;
79             VMint(k,j) = VMint(k,j) + tmp;
80         end
81         tmp1 = (-1)^(n+1)*2*sqrt(2)*alpha0int/(pi/2+n*pi)
82             *cos((pi/2+n*pi)*(t)).*Vk{j}(t,n)*dt;
83         Vkint(j) = Vkint(j) + tmp1;
84         VMint(j,j) = Vkint(j);
85     end
86 end
87 elapsedtime = toc;
88 fprintf('Assembly of q1 sub-vector code block execution time: %.4f ...
89         s\n', elapsedtime);
90
91 tic;
92 q1 = sum(VMint,2);
93 elapsedtime = toc;
94 fprintf('Generation of q1 code block execution time: %.4f s\n', ...
95         elapsedtime);
96
97 % ///// q2 /////
98 tic;
99 for j = 1 : M
100     tj1 = dt*(j-1);
101     tj = dt*j;
102     s = (tj-tj1)/2*s.ref + (tj+tj1)/2;
103     jac = (tj-tj1)/2;
104     for n = 0 : N
105         alpha0int = integral(@(x) alpha0(x, n), 0, 1);
106         for k = 1 : j-1
107             tmp = (-1)^(n+1)*2*sqrt(2)*alpha0int*(pi/2+n*pi)
108                 *sum(wt.*sin((pi/2+n*pi)*(s)).*VMc{k}(s,n))*jac;
109             VMcint(k,j) = VMcint(k,j) + tmp;
110         end
111         tmp1 = (-1)^(n+1)*2*sqrt(2)*alpha0int*(pi/2+n*pi)
112             *sum(wt.*sin((pi/2+n*pi)*(s)).*Vkc{j}(s,n))*jac;
113         Vkcint(j) = Vkcint(j) + tmp1;
114         VMcint(j,j) = Vkcint(j);
115     end
116 end
117 elapsedtime = toc;
118 fprintf('Assembly of q2 sub-vector code block execution time: %.4f ...

```

```

        s\n', elapsedtime);
117
118 tic;
119 q2 = sum(VMcint,2);
120 elapsedtime = toc;
121 fprintf('Generation of q2 code block execution time: %.4f s\n', ...
        elapsedtime);
122
123
124 %% ===== Q =====
125 % ///// Q1 /////
126 tic;
127 Q1 = zeros(M,M);
128
129 tic
130
131 for k = 1 : M
132     tk1 = (k-1)*dt;
133     tk = k*dt;
134     s = (tk-tk1)/2*s_ref + (tk+tk1)/2;
135     jac = (tk-tk1)/2;
136     Qeint = zeros(M,M);
137     tic
138     for i = 1:k-1
139         for j = 1:i-1
140             for n = 0 : N
141                 tmp2 = 2/(pi/2+n*pi)^2*sum(wt.*VM{i}(s,n)*(VM{j}(s,n)'))*jac;
142                 Qeint(i, j) = Qeint(i, j) + tmp2;
143             end
144             Qeint(j, i) = Qeint(i, j);
145         end
146         for n = 0 : N
147             tmp = 2/(pi/2+n*pi)^2*sum(wt.*VM{k}(s,n)*(VM{k}(s,n)'))*jac;
148             Qeint(i, i) = Qeint(i, i) + tmp;
149         end
150     end
151     for n = 0 : N
152         tmp = 2/(pi/2+n*pi)^2*sum(wt.*Vk{k}(s,n)*(Vk{k}(s,n)'))*jac;
153         Qeint(k, k) = Qeint(k, k) + tmp;
154     end
155     Q1 = Q1 + Qeint;
156
157     disp(['k = ', num2str(k)]);
158     elapsedtime = toc;
159     fprintf('Assembly of Qe{k} integral code block execution time: ...
        %.4f s\n', elapsedtime);
160 end
161 elapsedtime = toc;
162 fprintf('Assembly of Qeint{k} integral code block execution time: ...
        %.4f s\n', elapsedtime);
163

```

```

164 % ///// Q2 /////
165 tic;
166
167 Q2 = zeros(M,M);
168
169 tic
170 for k = 1 : M
171     tk1 = (k-1)*dt;
172     tk = k*dt;
173     s = (tk-tk1)/2*s_ref + (tk+tk1)/2;
174     jac = (tk-tk1)/2;
175     Qeint2 = zeros(M,M);
176     tic
177     for i = 1:k-1
178         for j = 1:i-1
179             for n = 0 : N
180                 tmp1 = 2*sum(wt.*VMc{i}(s,n)*(VMc{j}(s,n)'))*jac;
181                 Qeint2(i, j) = Qeint2(i, j) + tmp1;
182             end
183             Qeint2(j, i) = Qeint2(i, j);
184         end
185         for n = 0 : N
186             tmp = 2*sum(wt.*VMc{k}(s,n)*(VMc{k}(s,n)'))*jac;
187             Qeint2(i, i) = Qeint2(i, i) + tmp;
188         end
189     end
190     for n = 0 : N
191         tmp = 2*sum(wt.*Vkc{k}(s,n)*(Vkc{k}(s,n)'))*jac;
192         Qeint2(k, k) = Qeint2(k, k) + tmp;
193     end
194     Q2 = Q2 + Qeint2;
195
196     disp(['k = ', num2str(k)]);
197     elapsed_time = toc;
198     fprintf('Assembly of Qe2{k} integral code block execution time: ...
199             %.4f s\n', elapsed_time);
200 end
201 elapsed_time = toc;
202 fprintf('Assembly of Qeint{k} integral code block execution time: ...
203         %.4f s\n', elapsed_time);
204
205 % ///// Q3 /////
206 Q3 = dt*eye(M);
207
208 %% ==== W ====
209 % Calculate W1
210 W1 = 0;
211 for k = 1 : M
212     tk1 = dt*(k-1);
213     tk = dt*k;
214     s = (tk-tk1)/2*s_ref + (tk+tk1)/2;

```

```

213     jac = (tk-tk1)/2;
214     for n = 0 : N
215         alpha0int = integral(@(x) alpha0(x, n), 0, 1);
216         tmp = alpha0int^2*sum(wt.*cos((pi/2+n*pi)*(s)).^2)*jac;
217         W1 = W1 + tmp;
218     end
219 end
220
221 % Calculate W2
222 W2 = 0;
223 for k = 1 : M
224     tk1 = dt*(k-1);
225     tk = dt*k;
226     s = (tk-tk1)/2*s_ref + (tk+tk1)/2;
227     jac = (tk-tk1)/2;
228     for n = 0 : N
229         alpha0int = integral(@(x) alpha0(x, n), 0, 1);
230         tmp = ...
                (pi/2+n*pi)^2*alpha0int^2*sum(wt.*sin((pi/2+n*pi)*(s)).^2)*jac;
231         W2 = W2 + tmp;
232     end
233 end
234
235 % Formulate matrices Q, q, and W
236 Q = Q3 + Q1 + gamma*Q2;
237 q = q1 + gamma*q2;
238 W = W1 + gamma*W2;
239
240 % Solve quadratic optimization problem
241 U = quadprog(2*Q,q);
242
243 % Plot control w.r.t time
244 t = 0:dt:T;
245 figure
246 plot(t(1:end-1),U,"LineWidth",1)
247 xlabel('t', 'FontSize', 14);
248 ylabel('control', 'FontSize', 14);
249 filename = strcat('u_', num2str(T), '.mat');
250 save(filename,"U","t");
251
252 % Compute cost function
253 J = U'*Q*U + q'*U + W;
254
255 J = J/2;
256 ratio = J/T;
257
258 % Display results
259 fprintf(['T= ' num2str(T), ' J= ' num2str(J), ' Ratio=' ...
        num2str(ratio) '\n']);
260 end
261

```

```

262 function y = y0(x)
263     y = x;
264 end
265
266 function y = y1(x)
267     y = 0*x;
268 end
269
270 function phi = phin(x,n)
271     phi = sqrt(2).* sin((pi/2 + n * pi) .* x);
272 end

```

This is the code for the optimal control problems for one cycle.

```

1 function [J,u,y1r,y2r,y3r,z1r,z2r,z3r] = OptControl(eps)
2 % Check if input argument 'eps' is provided, otherwise set default value
3 if nargin == 0
4     eps = 0.5;
5 end
6
7 %% ==== setting Parameters =====
8 eps0 = 0.5;
9 epsmax = 1;
10
11 % The length of each edge.
12 L = 2;
13 Leps = [L-eps,L-eps,L-eps,eps,eps,eps];
14 Lopt = [L-eps0,L-eps0,L-eps0,eps0,eps0,eps0];
15 hmax = 0.01;
16
17 % Calculate number of intervals
18 l = L-epsmax;
19 v = 1;
20 N0l = ceil(l/hmax);
21 Nlz = ceil((epsmax-eps0)/hmax);
22 Nly = ceil((epsmax-eps)/hmax);
23
24 % Generate node coordinates
25 x0l = linspace(0,L-epsmax,N0l+1);
26 x1z = linspace(L-epsmax,L-eps0,Nlz+1);
27 x1y = linspace(L-epsmax,L-eps,Nly+1);
28
29 x1z(1) = [];
30 x1y(1) = [];
31
32 % node coordinates
33 xz = [x0l x1z];
34 xy = [x0l x1y];
35 % cell sizes
36 hz = diff(xz);
37 hy = diff(xy);

```

```

38 % number of cells
39 Nxy = length(hy);
40 Nxz = length(hz);
41
42 % cell indices for objective functional
43 E = 1:N0l+1;
44
45 % Parameters for small cycle
46 Neps = 10;
47 heps = eps/Neps;
48 hepsOpt = eps0/Neps;
49 xepsOpt = linspace(0,eps0,Neps+1);
50
51 % number of basis functions
52 Nfy = 2*(N0l+1);
53 Nfz = 2*(Nxz+1);
54 Nfeps = 2*(Neps+1);
55
56 %% compute z
57 Nz = [length(hz),length(hz),length(hz),Neps,Neps,Neps];
58 z = cell(6,1);
59 for i = 1 : 6
60     z{i} = zeros(2*Nz(i)+2,1);
61 end
62
63 % Solve equations for coefficients of z
64 syms a2z a3z a4z a5z a6z b1z b4z b5z b6z
65 eq1 = b1z - b5z == -v*Lopt(1);
66 eq2 = a4z*eps0 + b4z - b5z == 0;
67 eq3 = v + a4z - a5z == 0;
68 eq4 = a5z*eps0 + b5z - b6z == 0;
69 eq5 = a2z*Lopt(2) - b6z == 0;
70 eq6 = a2z + a5z - a6z == 0;
71 eq7 = b4z - a6z*eps0 - b6z == 0;
72 eq8 = a3z*Lopt(3) - b4z == 0;
73 eq9 = a3z + a6z - a4z == 0;
74 solu = solve(eq1,eq2,eq3,eq4,eq5,eq6,eq7,eq8,eq9);
75
76 az = [v; solu.a2z; solu.a3z; solu.a4z; solu.a5z; solu.a6z];
77 Bz = [solu.b1z; 0; 0; solu.b4z; solu.b5z; solu.b6z; ];
78
79 for i = 1 : 3
80     z{i}(1:Nz(i)+1) = az(i)*xz + Bz(i);
81     z{i}(Nz(i)+2:2*Nz(i)+2) = az(i)*hmax;
82 end
83
84 for i = 4 : 6
85     z{i}(1:Nz(i)+1) = az(i)*xepsOpt + Bz(i);
86     z{i}(Nz(i)+2:2*Nz(i)+2) = az(i)*hepsOpt;
87 end
88

```

```

89 %% boundary condition
90 Nf = [repmat(Nfy,1,6)];
91 index = cumsum(Nf);
92 index = [0 index];
93 for i = 1 : 3
94     % index of [y(0),y(L),y'(0),y'(L)];
95     bdy{i} = index(2*i-1)+[1,Nf(2*i-1)/2,Nf(2*i-1)/2+1,Nf(2*i-1)];
96     % index of [p(0),p(L),p'(0),p'(L)];
97     bdp{i} = index(2*i)+[1,Nf(2*i)/2,Nf(2*i)/2+1,Nf(2*i)];
98 end
99
100 %% FEM matrices
101 [As,Am] = graph1D.matirx_Hermite(N0l,x0l);
102
103 As_array = repmat({As}, 1, 6);
104 A = blkdiag(As_array{:});
105
106 A(Nfy+1:2*Nfy,1:Nfy) = Am;
107 A(3*Nfy+1:4*Nfy,2*Nfy+1:3*Nfy) = Am;
108 A(5*Nfy+1:6*Nfy,4*Nfy+1:5*Nfy) = Am;
109
110 A(1,Nfy+1) = A(1,Nfy+1)-1;
111
112 A(bdy{1}(2),bdy{1}(2)) = A(bdy{1}(2),bdy{1}(2)) - 2/(2*eps-3);
113 A(bdy{1}(2),bdy{2}(2)) = A(bdy{1}(2),bdy{2}(2)) + 1/(2*eps-3);
114 A(bdy{1}(2),bdy{3}(2)) = A(bdy{1}(2),bdy{3}(2)) + 1/(2*eps-3);
115
116 A(bdy{2}(2),bdy{1}(2)) = A(bdy{2}(2),bdy{1}(2)) + 1/(2*eps-3);
117 A(bdy{2}(2),bdy{2}(2)) = A(bdy{2}(2),bdy{2}(2)) - 2/(2*eps-3);
118 A(bdy{2}(2),bdy{3}(2)) = A(bdy{2}(2),bdy{3}(2)) + 1/(2*eps-3);
119
120 A(bdy{3}(2),bdy{1}(2)) = A(bdy{3}(2),bdy{1}(2)) + 1/(2*eps-3);
121 A(bdy{3}(2),bdy{2}(2)) = A(bdy{3}(2),bdy{2}(2)) + 1/(2*eps-3);
122 A(bdy{3}(2),bdy{3}(2)) = A(bdy{3}(2),bdy{3}(2)) - 2/(2*eps-3);
123
124 A(bdp{1}(2),bdp{1}(2)) = A(bdp{1}(2),bdp{1}(2)) - 2/(2*eps-3);
125 A(bdp{1}(2),bdp{2}(2)) = A(bdp{1}(2),bdp{2}(2)) + 1/(2*eps-3);
126 A(bdp{1}(2),bdp{3}(2)) = A(bdp{1}(2),bdp{3}(2)) + 1/(2*eps-3);
127
128 A(bdp{2}(2),bdp{1}(2)) = A(bdp{2}(2),bdp{1}(2)) + 1/(2*eps-3);
129 A(bdp{2}(2),bdp{2}(2)) = A(bdp{2}(2),bdp{2}(2)) - 2/(2*eps-3);
130 A(bdp{2}(2),bdp{3}(2)) = A(bdp{2}(2),bdp{3}(2)) + 1/(2*eps-3);
131
132 A(bdp{3}(2),bdp{1}(2)) = A(bdp{3}(2),bdp{1}(2)) + 1/(2*eps-3);
133 A(bdp{3}(2),bdp{2}(2)) = A(bdp{3}(2),bdp{2}(2)) + 1/(2*eps-3);
134 A(bdp{3}(2),bdp{3}(2)) = A(bdp{3}(2),bdp{3}(2)) - 2/(2*eps-3);
135
136 b = zeros(size(A,1),1);
137 b(1) = b(1)-v;
138 b(Nfy+1:2*Nfy) = Am*z{1}([E,E+Nfz/2]);
139 b(3*Nfy+1:4*Nfy) = Am*z{2}([E,E+Nfz/2]);

```

```

140 b(5*Nfy+1:6*Nfy) = Am*z{3}([E,E+Nfz/2]);
141
142 %% boundary condition
143 % y2(0) = 0;
144 A(bdy{2}(1,:),:) = 0;
145 A(bdy{2}(1),bdy{2}(1)) = 1;
146 b(bdy{2}(1)) = 0;
147
148 % y3(0) = 0;
149 A(bdy{3}(1,:),:) = 0;
150 A(bdy{3}(1),bdy{3}(1)) = 1;
151 b(bdy{3}(1)) = 0;
152
153 % p2(0) = 0;
154 A(bdp{2}(1,:),:) = 0;
155 A(bdp{2}(1),bdp{2}(1)) = 1;
156 b(bdp{2}(1)) = 0;
157
158 % p3(0) = 0;
159 A(bdp{3}(1,:),:) = 0;
160 A(bdp{3}(1),bdp{3}(1)) = 1;
161 b(bdp{3}(1)) = 0;
162
163 % Solve linear system
164 solu = A\b;
165
166 for i = 1 : 3
167     y{i} = solu(bdy{i}(1):bdy{i}(2));
168     p{i} = solu(bdp{i}(1):bdp{i}(2));
169     dy{i} = solu(bdy{i}(3):bdy{i}(4))/hmax;
170     dp{i} = solu(bdp{i}(3):bdp{i}(4))/hmax;
171 end
172
173 % Compute objective function J
174 J = 0;
175 for i = 1 : 3
176     tmp = norm(y{i}(E)-z{i}(E))^2*hmax;
177     J = J + tmp;
178 end
179
180 % Compute control variable u
181 u = v - p{1}(1)
182 J = J + (u-v)^2;
183 J = J/2
184
185 y1r = y{1}(E);
186 y2r = y{2}(E);
187 y3r = y{3}(E);
188
189 z1r = z{1}(E);
190 z2r = z{2}(E);

```

```
191  z3r = z{3}(E);  
192  end
```

A.4 Differentiability Properties

Define the function

$$\mathcal{H}(t) = \int_0^t u(s) \sin \left(\left(\frac{\pi}{2} + n\pi \right) \frac{t-s}{t_0} \right) ds$$

and for $h > 0$

$$\begin{aligned} \text{Diff}_h \mathcal{H}(t) &= \frac{\mathcal{H}(t+h) - \mathcal{H}(t)}{h} \\ &= \frac{1}{h} \int_t^{t+h} u(s) \sin \left(\left(\frac{\pi}{2} + n\pi \right) \frac{t+h-s}{t_0} \right) ds \\ &\quad + \frac{1}{h} \int_0^t u(s) \left[\sin \left(\left(\frac{\pi}{2} + n\pi \right) \frac{t+h-s}{t_0} \right) - \sin \left(\left(\frac{\pi}{2} + n\pi \right) \frac{t-s}{t_0} \right) \right] ds. \end{aligned}$$

Hence we have

$$\begin{aligned} &\left| \text{Diff}_h \mathcal{H}(t) - \int_0^t u(s) \frac{\frac{\pi}{2} + n\pi}{t_0} \cos \left(\left(\frac{\pi}{2} + n\pi \right) \frac{t-s}{t_0} \right) ds \right| \\ &\leq \int_t^{t+h} |u(s)| \left| \frac{\sin \left(\left(\frac{\pi}{2} + n\pi \right) \frac{t+h-s}{t_0} \right)}{h} \right| ds \\ &\quad + \left| \int_0^t u(s) \left[\frac{\sin \left(\left(\frac{\pi}{2} + n\pi \right) \frac{t+h-s}{t_0} \right) - \sin \left(\left(\frac{\pi}{2} + n\pi \right) \frac{t-s}{t_0} \right)}{h} - \frac{\frac{\pi}{2} + n\pi}{t_0} \cos \left(\left(\frac{\pi}{2} + n\pi \right) \frac{t-s}{t_0} \right) \right] ds \right| \\ &\leq \int_t^{t+h} |u(s)| \frac{\frac{\pi}{2} + n\pi}{t_0} ds \\ &\quad + \left| \int_0^t u(s) \frac{1}{h} \left[\int_t^{t+h} \frac{\frac{\pi}{2} + n\pi}{t_0} \left[\cos \left(\left(\frac{\pi}{2} + n\pi \right) \frac{\tau-s}{t_0} \right) - \cos \left(\left(\frac{\pi}{2} + n\pi \right) \frac{t-s}{t_0} \right) \right] d\tau \right] ds \right| \\ &\leq \frac{\frac{\pi}{2} + n\pi}{t_0} \int_t^{t+h} |u(s)| ds + h \frac{(\frac{\pi}{2} + n\pi)^2}{t_0^2} \int_0^t |u(s)| ds. \end{aligned}$$

Thus $\mathcal{H}(t)$ is differentiable almost everywhere and thus absolutely continuous on $[0, T]$ see [72], Theorem 14. Almost everywhere on $[0, T]$ we have the derivative

$$\mathcal{H}'(t) = \frac{\frac{\pi}{2} + n\pi}{t_0} \int_0^t u(s) \cos \left(\left(\frac{\pi}{2} + n\pi \right) \frac{t-s}{t_0} \right) ds.$$

This implies in particular that $\alpha'_n(t) \in L^2(0, T)$.

Theorem 3.2 in [22] contains a traveling wave representation of the state.