

October 4, 2024

Professor Jan Owsiniński
Z-ca Dyrektora d/s Naukowych
Instytut Badań Systemowych
Polskiej Akademii Nauk

Distinguished Professor Owsiniński,

I am presenting for your consideration the review of the dissertation

OUTER APPROXIMATION SCHEME FOR WEAKLY CONVEX
CONSTRAINED OPTIMIZATION PROBLEMS

by Giovanni Brucola

The thesis delves into the analysis of constrained optimization problems with weakly convex constraint functions and objective functions. This is a crucial and contemporary research topic due to the prevalence of non-convex and non-smooth optimization problems in various applications, including data science, finance, and optimal control. The severe theoretical and computational challenges these problems present necessitate in-depth theoretical studies and the development of new algorithmic approaches. The thesis is a significant contribution to the field.

The main results of the thesis are contained in Chapters 2 through 6. In Chapter 2, the author studies subdifferentials and ε -subdifferentials of weakly convex functions. An important theoretical contribution is Theorem 15, which extends the classical Moreau-Rockafellar Theorem to the case of proximal ε -subdifferentials. A further significant contribution is Proposition 19. An interesting research question arising in the context of the analysis provided in this chapter is the case of an integral of weakly convex functions rather than just a sum. It is well beyond the scope of this dissertation, but it is essential for stochastic optimization problems involving such functions, and I recommend the author look into it in the future.

Chapter 3 contains some of the main theoretical and algorithmic contributions of the thesis: the cutting spheres method and its convergence analysis. An interesting observation is Theorem 20 stating that problems with weakly convex functions can be equivalently formulated as projection problems on sets defined by weakly convex constraints. The remaining analysis exploits this transformation and addresses the problem by constructing quadratic cuts separating infeasible points from the feasible set. The central results of the chapter are the Cutting Spheres Algorithm of Section 3.4 and its convergence analysis in Section 3.5, Theorem 30. It is a precise and convincing contribution. As is usually the case, after a result is obtained, other approaches offer themselves. In this case, the analysis could be streamlined along the following lines.

Let $K(\varepsilon)$ be the set of iterations k such that at least one constraint is violated by ε or more. If k is in $K(\varepsilon)$, the cut constructed at k guarantees that all future points are at least ε/C away from the

kth point, where C is a universal constant. As all points are in a ball of radius equal to the distance to the optimal point, the set $K(\epsilon)$ is finite for any $\epsilon > 0$. Consequently, the violation of the constraints converges to 0. The author may explore this line of argument in his further research of methods using separating surfaces.

Chapter 4 analyzes the outer approximating problem in the Cutting Spheres Method, a nonconvex quadratically constrained quadratic optimization problem. Much of it presents existing results systematically. However, I noticed a very interesting original result in Theorem 57, which proves the convexity of the generalized image set. Its statement is surprising, and the proof, exploiting the fact that the dimension is larger than the number of constraints, is very clever.

Chapter 5 is devoted to solution methods for the outer approximating quadratic problem. As the problem is non-convex by construction, highly specialized approaches must be developed to overcome this difficulty. Here, hidden convexity plays a key role. However, using it would require dropping cuts to keep the number of cuts below the dimension. The author may explore these issues in his further research. On the other hand, I would not be very concerned with NP-hardness; the most interesting problems are NP-hard.

Chapters 6 and 7 present details of the computational realization and an application to the circular packing problem. They are essential for the dissertation because they demonstrate that the Author's theoretical ideas can lead to an effective computational method. Regarding the packing example, I want to notice that the original problem is already a quadratically constrained quadratic optimization problem, so the only simplification achieved is reducing the number of constraints in the “master” problem.

Summing up, the thesis is an excellent contribution to the theory and methods of nonsmooth and nonconvex optimization. It contains profound results with nontrivial proofs and provides an effective method for solving a difficult class of such problems. It certainly fulfills the requirements for a doctoral dissertation; I do not doubt it would be highly evaluated at my university and other leading universities worldwide.

Sincerely yours,



Andrzej Ruszczyński

Board of Governors Professor

Tel: 848-445-3184

E-mail: rusz@business.rutgers.edu

P.S.: I noticed a small number of typos and minor inaccuracies, which in no way diminish the quality of the dissertation. I communicated these typos directly to the Author.