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Report on the PhD Thesis

“Outer Approximation Scheme for Weakly Convex Constrained Optimization Problems”

**submitted by
Giovanni Bruccola
(Systems Research Institute Polish Academy of Sciences)**

The PhD thesis of Giovanni Bruccola proposes an outer approximation scheme for solving weakly convex constrained optimization problems. Motivated by the role played by quadratic terms in the characterization of weakly convex functions, the idea is to perform the outer approximation by using quadratic instead of linear cuts, thus solving a quadratically constrained quadratic problem (QCQP) at each iteration. Under certain assumptions, necessary and sufficient conditions are formulated to characterize global optimal solutions of such a problem. A convergence analysis for the outer approximation scheme is performed, which shows that every generated convergent subsequence generated converges to a global minimum of the general weakly convex optimization problem. The numerical performance of the algorithm is illustrated in the context of solving the circular packing problem, which can be reformulated as a weakly convex constrained optimization problem.

The doctoral thesis of Giovanni Bruccola has 158 pages and is divided into six chapters besides the introductory chapter and the bibliography. It is based on 4 research articles of the doctoral candidate, one published in *Optimization*, one in the proceeding of the 2023 European Control Conference, another submitted to a journal and the fourth in preparation.

Structure and contents of the PhD thesis

The introductory chapter introduces the main object of investigation, the weakly convex constrained optimization problem, and provides a motivation for performing quadratic cuts in the outer approximation scheme. Several topological notions for sets and functions are reviewed. Finally, a comprehensive literature review on outer approximation algorithms for convex and nonconvex optimization problems is given.



Chapter 2 is devoted to the notion of weak convexity and various related properties. Several subdifferential notions are introduced, whereby the Φ -subdifferential plays an essential role in the formulation of the quadratic cuts. Calculus rules for proximal subdifferential are also proven, mainly relying on such existing for convex subdifferentials. Inexact proximal maps are also introduced and characterized by means of inclusions expressed in terms of proximal ε -subdifferentials.

At the beginning of Chapter 3, it is shown that under mild assumptions, the general weakly convex constrained optimization problem can be seen as equivalent to minimizing of the normed squared with respect to weakly convex inequality constraints. The latter problem is much more convenient for designing a cutting sphere algorithm based on an outer approximation technique. The outer approximation of the feasible set of the latter problem is rigorously described, as are the convergence properties of the cutting sphere algorithm. The theoretical results, the proofs of which are not very involved, are illustrated by examples with figures that give a useful geometric interpretation of the outer approximation approach.

In Chapter 4, the outer approximation problem is studied from the perspective of a QCQP problem, more precisely, of a scalar-quadratically constrained quadratic programs (S-QCQP), as introduced by the candidate and his E. Bedanrczuk. The main results in the literature on QCQP and the relationships between these results are highlighted. The concept of hidden convex QCQP is also introduced and the connections between hidden convexity, convex relaxations and Lagrangian duality are explored. Furthermore, QCQP problems in Hilbert spaces are considered and a global optimality for these problems is characterized in terms of necessary and sufficient KKT optimality conditions obtained as a consequence of the S-Lemma. Finally, it is shown that for S-QCQP in Hilbert spaces, the Generalized Image Set (GIS) is convex, so one can formulate global optimality via KKT conditions using the S-Lemma.

In Chapter 5, the results of the previous chapter are applied to the outer approximation problem - connections between this problem and the feasibility problem between a sphere and a polyhedron are worked out. It is also shown that the feasibility problem between a sphere and a polyhedron can be rewritten as a hidden convex QCQP, which is generally NP-hard.

Chapter 6 describes two computational realizations of the cutting sphere algorithm. The warm restart version has weaker convergence properties than the standard one, but is more practical for applications. In particular, the number of constraints in the outer approximation problems solved at each iteration remains limited. The theoretical properties of the algorithm with warm restarts are analyzed when the method terminates in a finite number of steps and when it does not terminate in a finite number of steps. The inexact cutting sphere algorithm, a modification of the cutting sphere algorithm with warm restarts, ensures that the complexity bound for solving an outer approximation step does not grow exponentially with the number of constraints.

In Chapter 7, the numerical experiments are carried out on the packing problem, which is reformulated as a weakly convex constrained optimization problem. The experiments are not very comprehensive and comparisons with other methods designed to solve the same problem are not provided.



Assessment of the thesis

The thesis of Mr. Giovanni Brucola's thesis is a nice piece of research that fulfils all the expectations of a mathematics PhD thesis. It aims at developing a numerical method for solving a certain class of optimization problems - for this purpose, theoretical investigations on weakly convex functions and appropriate subdifferential notions are carried out, an outer approximation scheme is formulated, the global optimality of the resulting QCQP problems is studied, the convergence properties of the method are analyzed, and finally computational realisations of the algorithm and numerical experiments are discussed.

The thesis shows that the candidate has a good knowledge and understanding of different areas of mathematics, such as optimization, convex and nonsmooth analysis, and numerical analysis. Unfortunately, the thesis contains some inaccuracies, as discussed below. Nevertheless, I recommend the thesis of Mr Giovanni Brucola for acceptance.

Yours sincerely,

Radu Bot

Minor remarks:

- page 1, problem (P0): how is this problem defined for $n=1$?
- page 2: the statement which refers to (1.2) seems to be incomplete;
- page 4: every empty set is an affine subspace;
- page 10 and not only: it should be *Hilbert spaces*, not just *Hilbert*;
- page 10, Proposition 4: since ρ is fixed, the statement "if there exists ..." is misleading;
- page 11: how is the subdifferential defined at points which are not in the domain of f ?
- page 12: what is the meaning of "add corollary 16.49"?
- page 12, Corollary 7: the result seems to be true for each $a \geq \rho/2$;
- page 12: formula (2.4) is confusing as it is not clear how the inequality is meant: is φ fixed or should the inequality be fulfilled for every element in Φ_a ? What is the meaning of Φ ?
- page 14: what is the role played by Φ in (2.7)? The subdifferential depends only on a and ε ;
- page 15: formula (2.11) is confusing; should φ not have a lower index i ?
- page 15, Proposition 11: $\subset$ is the symbol for strict inclusion. Why should the inclusions be strict?
- page 16, Definition 12: what is h ?
- page 17: the continuity (Rockafellar) regularity condition and the Attouch-Brezis one are not really comparable, since they assume different topological properties for the two functions;
- page 18, formula (2.16): I would say that the statement is true everywhere and not only



on the domain of the conjugate of the sum;

- page 19, Theorem 15: for the equality (2.22) one should assume that either f_0 or f_1 is continuous at a point in the intersection of their domains;
- page 20, formula (2.30) and also later: the subdifferential with lower index $(2, \rho_0/2)$ has not been introduced in the thesis;
- page 21, Theorem 16: it should be better explained where the non emptiness of the subdifferential is needed;
- page 35, Lemma 23: one should recall the definition of φ ;
- page 42, Remark 11: in Hilbert spaces one would to adapt Theorem 29, for instance, by taking into account that bounded sequences have weakly convergent subsequences.