

OUTER APPROXIMATION SCHEME FOR WEAKLY CONVEX
CONSTRAINED OPTIMIZATION PROBLEMS

by

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“Freedom is the freedom to say that two plus two makes four. If that is granted,
all else follows.”

-George Orwell, 1984.

Abstract

Outer approximation methods have been used to solve linear programming problems in the 60s and are still employed to solve variational inequalities problems, general convex problems, mixed integer linear and nonlinear programming problems.

In this work, we propose a novel outer approximation scheme for solving weakly convex constrained optimization problems. The main idea, motivated by the properties of weakly convex functions, is to use quadratic cuts instead of linear cuts and solve at each iteration an outer approximation problem in the form of QCQP problem. We investigate Lagrangian duality and convex relaxation for non-convex QCQP. We show that the outer approximation problem belongs to the class of S-QCQP, where the quadratic terms of the objective function and the constraints are formed by a (possibly negative) scalar multiplied by the normed squared. We establish necessary and sufficient condition for global optimality of S-QCQP, under certain assumptions. The main result of this work is that every convergent subsequence produced by the outer approximation scheme proposed converges to the global minimum of a general weakly convex optimization problem.

The algorithm is applied to the circular packing problem, which can be reformulated as a weakly convex constrained problem.

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Figure 1: Logo of our project

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Symbols

H Hilbert space.

$\mathbb{R}^n$ n -dimensional Euclidean space.

$\|\cdot\|$ norm of the space.

$\langle \cdot, \cdot \rangle$ inner product.

$\text{int } C$ the interior of a set $C \subset H$.

$\text{ri } C$ the relative interior of a set $C \subset H$.

$\text{aff } C$ the affine hull interior of a set $C \subset H$.

$B(x, \delta)$ the open ball in H with centre x and radius $\delta > 0$.

$\overline{B(x, \delta)}$ the closure of $B(x, \delta)$

$\text{dom } f$ the domain of a function f .

$\text{lev}_{\leq \alpha} f(x)$ lower level set of a function $f : H \rightarrow (-\infty, +\infty]$ at $\alpha \in \mathbb{R}$.

$\Gamma_0(H)$ the set of functions $f : H \rightarrow (-\infty, +\infty]$ which are proper, lower semicontinuous
([lsc](#)) and convex.

$\Gamma_\rho(H)$ the set of functions $f : H \rightarrow (-\infty, +\infty]$ which are proper, [lsc](#) and ρ -weakly convex.

$\partial_0 f(x)$ convex subdifferential of a function f at $x \in H$.

$\partial_\Phi^a f(x)$ subdifferential of a function f at $x \in H$ as defined in [\(8\)](#).

φ elementary function belonging to the class Φ^a defined in [\(8\)](#).

$\partial_a f(x)$ proximal subdifferential of a function f at $x \in H$.

P_C projection operator of a point in H to a convex set $C \subset H$.

$prox_f$ proximal point operator $prox_f : H \rightarrow H$, $prox_f := \arg \min_{y \in H} \{f(y) + (1/2)\|\cdot - y\|^2\}$.

$I(\bar{x})$ set of indices of the constraints of [\(P\)](#) which are not satisfied at $\bar{x} \in H$, see [\(3.11\)](#).

$L_i(\varphi_i, \bar{x})(x)$ the quadratic cuts defined in [\(2.12\)](#). They are constructed starting from a point $\bar{x} \in H$, with $i \in I(\bar{x})$.

Acronyms

lsc lower semicontinuous

QCQP Quadratically Constrained Quadratic Programs

S-QCQP Scalar-Quadratically Constrained Quadratic Programs

GIS Generalized Image Set

CPP Circular Packing Problem

Chapter 1

Introduction

1.1 Scope of the thesis

We consider the following problem:

$$\underset{x \in \mathbb{R}^{n-1}}{\text{minimize}} \quad F(x) \quad \text{s.t.} \quad g_i(x) \leq 0, \quad i \in \{2, \dots, m\}, \quad (\text{P0})$$

where $n \in \mathbb{N}^*$, $F: \mathbb{R}^{n-1} \rightarrow (-\infty, +\infty]$ is a proper lower-semicontinuous (lsc) β -weakly convex function with $\beta_1 \in \mathbb{R}_+$ and, for every $i \in \{2, \dots, m\}$, $g_i: \mathbb{R}^{n-1} \rightarrow (-\infty, +\infty]$ is a proper lsc β_i -weakly convex function with $\beta_i \in \mathbb{R}_+$. This problem can be reformulated as

$$\underset{x \in S}{\text{minimize}} \quad F(x) \quad (1.1)$$

where $S = \bigcap_{i=2}^m \text{lev}_{\leq 0} g_i$ and, for every $i \in \{1, \dots, m\}$, $\text{lev}_{\leq 0} g_i = \{x \in \mathbb{R}^{n-1} \mid g_i(x) \leq 0\}$ is the lower-level set of g_i at height 0.

Problem (1.1) defines a superclass of convex optimization problems, due to the fact that any convex function is weakly convex with $\beta = 0$. Many optimization

problems can be cast in the form of (P0), e.g. every QCQP problem is a subclass of (P0).

In Chapter 3, Section 3.2, we show that, under mild assumptions, (P0) can be rewritten as follows:

$$\underset{x \in \mathcal{A}}{\text{minimize}} \quad J(x) = \frac{1}{2} \|x - z\|^2, \quad (\text{P})$$

where $z \in \mathbb{R}^n$ and

$$\mathcal{A} := \{x \in \mathbb{R}^n \mid (\forall i \in I = \{1, \dots, m\}) f_i(x) \leq 0\}.$$

and for every $i \in I$, function $f_i: \mathbb{R}^n \rightarrow (-\infty, +\infty]$ is proper, lsc, and ρ_i -weakly convex with $\rho_i \in \mathbb{R}_+$.

Due to this fact, in the thesis we consider problem (P). Let J^* be the optimal value of (P). If, for every $i \in I$, f_i is continuous on a lower level set

$$\text{lev}_{\leq \alpha} J = \{x \in \mathbb{R}^n \mid J(x) \leq \alpha\}, \quad (1.2)$$

and the height $\alpha > J^*$.

For solving problem (P) we propose a new outer approximation method called *cutting sphere method*. Cutting sphere method rely on constructing a sequence of quadratically constrained quadratic problem (QCQP) solutions of which approximate solutions to problem (P). The quadratic constraints of (QCQP) relax the original constraints of (P). A number of computationally realizable variants of the generic cuuting spheres algorithm is also considered.

At each step k , we solve a quadratically constrained (possibly nonconvex) optimization problem, which is an outer approximation to the original problem (P), in the sense that the set $\mathcal{A}$ is replaced by a superset $\mathcal{O}_k$, i.e.

$$A \subset \mathcal{O}_k,$$

see Proposition 11.

The main result of this work is that the every convergent subsequence generated by the cutting sphere method converges to a global minimum of (P), see Theorem 30.

1.1.1 Motivation

Our motivation is to provide an algorithm for constrained optimization problem (P0), which makes a direct use of weak convexity of the functions involved.

In contrast to the convex case, a conceivable passage from the constrained problems (P0) and (P) to the unconstrained ones which relies on adding the indicator function of the set $\mathcal{A}$ to the objective F is not that easily realizable mainly due to the fact that the relationships between weak convexity of a function, weak convexity of its level sets and/or weak convexity of the indicator function of $\mathcal{A}$ are still an active research topic and different concepts of a weakly convex set are taken into account (see Balashov and Repovš (2010); Lopushanski (2018)).

Weakly convex constrained problems can be approached by quadratic regularization techniques as described in Boob et al. (2023); Ma et al. (2020) where authors solve a sequence of strongly convex problems, however these papers results provide

only convergence to nearly stationary and ε -feasible solutions.

1.2 Notation and Preliminaries

Given two set $\mathcal{K}$ and $\mathcal{C}$, $|\mathcal{K}|$ denotes the cardinality of $\mathcal{K}$, while $\mathcal{K} \setminus \mathcal{C}$ denotes the set $\{k \in \mathcal{K} \mid k \notin \mathcal{C}\}$.

Let H be a Hilbert space with norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$. A Hilbert space is a uniformly convex Banach space, since the following result holds.

Proposition 1. ([Bauschke and Combettes, 2017b](#), Corollary 2.15) Let $x, y \in H$ and $\lambda \in \mathbb{R}$. Then,

$$\|\lambda x + (1 - \lambda)y\|^2 + \lambda(1 - \lambda)\|x - y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2.$$

We utilize Proposition 1 in some key moments in the following chapters.

We say that C is an affine subspace if $C \neq \emptyset$ and $\forall \lambda \in \mathbb{R}$ and for all distinct $x, y \in C$

$$\lambda x + (1 - \lambda)y \in C$$

$\text{aff} C$ denotes the affine hull of C , i.e. the smallest affine subspace of H containing C , $\text{int} C$ denotes the interior of C ,

$$\text{int} C := \{x \in C \mid (\exists \epsilon > 0) (x + \epsilon B) \subset C\}.$$

$B := \{x \mid \|x\| \leq 1\}$ be the *unit ball* in H . The relative interior of C , denoted $\text{ri} C$,

can be expressed in $\mathbb{R}^n$ as :

$$ri\,C := \{x \in aff\,C \mid (\exists \epsilon > 0) (x + \epsilon B) \cap (aff\,C) \subset C\}. \quad (1.3)$$

A function $f : H \rightarrow (-\infty, +\infty]$ is proper if

$$\text{dom } f := \{x \in H \mid f(x) < +\infty\} \neq \emptyset.$$

Let $\mathcal{V}(x_0)$ be the family of all neighborhoods of $x_0 \in H$ in the norm topology, f is lower semicontinuous (*lsc*) at x_0 , if

$$(\forall \xi \in (-\infty, f(x_0))) \quad (\exists V \in \mathcal{V}(x_0)) \quad f(x_0) \in (\xi, +\infty], \forall x_0 \in V.$$

If f is *lsc* at every $x \in H$, we say that f is **lsc**.

If a function $f : H \rightarrow (-\infty, +\infty]$ is proper, **lsc** and convex, we say $f \in \Gamma_0(H)$.

We denote open intervals with round brackets and close intervals with square brackets. For $\alpha \in \mathbb{R}$, we denote the level sets as $\text{lev}_{=\alpha} f := \{x \in H \mid f(x) = \alpha\}$ and $\text{lev}_{\leq \alpha} f := \{x \in H \mid f(x) \leq \alpha\}$.

1.3 Outer approximations algorithms for convex optimization

Consider the following convex constrained problem

$$\min_{x \in H} G(x) \quad \text{s.t. } x \in C, \quad (\text{CP})$$

where $G \in \Gamma_0(H)$ and C is convex. Outer approximation methods for convex optimization, see e.g. [Combettes \(2000\)](#), [Combettes and Pesquet \(2004\)](#), [Drori and Teboulle \(2016\)](#), generalize the cutting plane method introduced by [Kelley \(1960\)](#). The main idea of the cutting plane method for (CP) is to generate, at each step k , hyperplanes which separate the current iterate x_k from the feasible set C . In this way, the hyperplanes generate an outer approximation set of C , which does not contain the current iteration x_k . By minimizing the objective function of (CP) over the newly generated outer approximation set, we find the new iterate x_{k+1} by approaching in this way the feasible set C . We can say that the hyperplanes *cut* the current iteration x_k .

It is important to notice the connection between the cutting plane method and monotone operator theory. A *cutter* is an operator $T : H \rightarrow H$ such that

$$\text{Fix } T \subseteq h(x, Tx) := \{u \in H \mid \langle u - Tx, x - Tx \rangle \leq 0\}, \forall x \in H, \quad (1.4)$$

where $\text{Fix } T := \{x \in H \mid Tx = x\}$. Then, for a fixed $k \in \mathbb{N}$, the set bounding hyperplane of $h(x_k, Tx_k)$ *cuts* the space in two half-spaces, one which contains $\text{Fix } T$ and the other which contains x_k . The name cutter operators was suggested by [Cegielski and Censor \(2011\)](#), but the class of operators satisfying (1.4) was introduced, ante litteram, by [Combettes \(2001\)](#) and also studied, for example, in [Zaknoon \(2003\)](#); [Censor and Segal \(2009\)](#) under different names. The goal of the papers cited above is to propose an algorithm that, given a sequence of cutter operators $\{Tk\}_{k \in \mathbb{N}}$, $Tk : H \rightarrow H$, converges to a common fixed point.

Finally, also bundle methods, e.g. [Kiwiel \(1990\)](#); [Lukšan and Vlček \(1999\)](#), can

be seen as a generalization of the cutting plane algorithm by Kelley (1960), since the bound produced at every iteration consists in a hyperplane.

1.3.1 Outer Approximation in nonconvex settings and Contribution

Several outer approximation methods deal with nonconvex constrained optimization problems. For example, Yamada et al. (2010), Bienstock et al. (2019), Tuy and Van Thuong (1988), minimize the objective over the intersection $C \cap S$ of a convex set C and a non convex set S having a specific structure. The core idea is to move in the direction of C with a classic convex outer approximation method. In order to converge to a point in $C \cap S$, *ad hoc* techniques are employed.

On the other hand, the core idea of our method differs from the other outer approximation methods that we can find in the literature, and it is briefly explained in the following.

It doesn't always exist a hyperplane separating the feasible set of (P), $\mathcal{A}$, to a point $x \in H \setminus \mathcal{A}$.

Hence, we replace hyperplanes with spheres by using the tools provided by abstract convexity.

In the setting of Abstract convexity (which include weak convexity), for unconstrained optimization, a generalized cutting plane method can be found in Pallaschke and Rolewicz (2013); Rubinov (2013), where successive underestimations of the abstract convex objective function are found by using the Φ -subgradient inequality (see (2)) instead of the convex subgradient.

We apply an analogous underestimation to the weakly convex functions that

defined the constraint set $\mathcal{A}$ of Problem (P), in order to generate a new outer approximation set at each iteration. The resulting outer approximation set is the complement of union of open balls.

The iterate x_k lies in the interior of the union of balls, which is cut out (in the sense that it is infeasible for the outer approximation set) while $\mathcal{A}$ lies in the complement of the union of open balls. Our original way to construct the outer approximation set is formally stated in Chapter 2, Section 2.4 and Chapter 3, Section 3.3. In Subsection 3.3.1, a formal geometrical interpretation of our outer approximation set is given.

At each iteration, we solve a polynomial quadratic optimization problem. We call our method *cutting spheres algorithm 1*. We prove that, for the cutting spheres algorithm 1 operating with cumulative basis (see Combettes (2000)) there exists a convergent subsequence and every convergent subsequence converges to the global optimal solution of (P), see Section 3.4.

The drawback of using cumulative basis is that the number of constraints of the outer approximation problem can go to infinity as the number of iteration increases. Moreover, the outer approximation problem that we solve at every iteration is NP-Hard. That is why, in Chapter 6, we propose other two algorithms, respectively the cutting sphere with warm restarts Algorithm 2 and the inexact cutting sphere Algorithm 3, which sacrifice some convergence properties of the cutting spheres algorithm 1, but are more suitable for practical applications.

We apply our algorithms to the Packing problem, see Chapter 7.

Chapter 2

Weak convexity

2.1 Chapter Outline

- In Section 2.2, we define weak convexity and recall some basic properties.
- In Section 2.3, we provide some useful tools in the analysis of weakly convex functions: the proximal subdifferential and the Φ -subdifferential.
- In Section 2.4, we present the core idea of this work. Given a set described by weakly convex constraints, we exploit the Φ -subdifferential to build an outer approximation set.
- In Section 2.5, we prove sum rules for proximal subdifferentials in the presence of weak convexity.
- In Section 2.6, we provide inexact proximal subdifferentials for weakly convex functions.

2.2 Definition

We start with the formal definition of ρ -weak convexity.

Definition 2 ((Cannarsa and Sinestrari, 2004, Proposition 1.1.3)). Let H be a Hilbert space. We say that a function $f : H \rightarrow (-\infty, +\infty]$ is ρ -weakly convex if $f(\cdot) + \frac{\rho}{2} \|\cdot\|^2$ is convex.

Consider the following definition of ρ -paraconvexity (a.k.a. ρ -semiconvexity, see Cannarsa and Sinestrari (2004)) in Hilbert.

Definition 3 (Jourani (1996); Rolewicz (1979)). Let H be a Hilbert space. We say that a function $f : H \rightarrow (-\infty, +\infty]$ is Paraconvex if there exists $\rho \geq 0$ such that for every $\lambda \in (0, 1)$, $(\forall (x, y) \in H^2)$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) + \frac{\rho}{2} \lambda(1 - \lambda) \|x - y\|^2. \quad (2.1)$$

In the next proposition, we show that the definitions of weak convexity and paraconvexity coincide in Hilbert, i.e. (2.1) holds with $\rho \geq 0$ if and only if f is ρ -weakly convex.

Proposition 4. Let H be a Hilbert space. A function $f : H \rightarrow (-\infty, +\infty]$ is ρ -weakly convex if and only if there exists a positive constant $\rho \geq 0$ such that for every $\lambda \in (0, 1)$, $(\forall (x, y) \in H^2)$, the following inequality holds.

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) + \frac{\rho}{2} \lambda(1 - \lambda) \|x - y\|^2. \quad (2.2)$$

Proof. By Definition 2, if f is ρ -weakly convex, $f + (\rho/2)\|\cdot\|^2$ is convex, i.e.

$$\begin{aligned} (f + \frac{\rho}{2}\|\cdot\|^2)(\lambda x + (1 - \lambda)y) &\leq \lambda(f(x) + \frac{\rho}{2}\|x\|^2) + (1 - \lambda)(f(y) + \frac{\rho}{2}\|y\|^2) \\ &= \lambda f(x) + (1 - \lambda)f(y) + \frac{\rho}{2}(\lambda\|x\|^2 + (1 - \lambda)\|y\|^2) = \\ &\lambda f(x) + (1 - \lambda)f(y) + \frac{\rho}{2}(\|\lambda x + (1 - \lambda)y\|^2 + \lambda(1 - \lambda)\|x - y\|^2), \end{aligned}$$

where the last equality holds by Proposition 1. $\square$

Remark 1. It would be impossible to apply Proposition 1 in a non uniformly convex Banach space and Proposition 1 is essential for completing the proof of Proposition 4. A similar result, in Hilbert spaces, can be found in Rolewicz (2005). In $\mathbb{R}^n$, an alternative proof of Proposition 4 is found in (Cannarsa and Sinestrari, 2004, Proposition 1.1.3).

We define $\Gamma_\rho(H)$ as the set of functions $f : H \rightarrow (-\infty, +\infty]$ which are proper, lsc and ρ -weakly convex.

2.3 Subdifferentials of weakly convex functions

The subdifferential at a point $x \in H$, in the sense of convex analyses, is a set defined as follows.

Definition 5. (Bauschke and Combettes, 2017b, Definition 16.1) Let $f : H \rightarrow (-\infty, +\infty]$ be proper.

$$\partial_0 f(\bar{x}) := \{u \in H \mid (\forall x \in H) \ f(x) - f(\bar{x}) \geq \langle u, x - \bar{x} \rangle\}. \quad (2.3)$$

Notice that, by bringing the term $f(\bar{x})$ in the right-hand side of (2.3), we get a continuous affine minorant of f . More precisely, the following result holds. (add corollary 16.49)

Theorem 6 (Bauschke and Combettes (2017a), Theorem 9.23). *Let $f : H \rightarrow (-\infty, +\infty]$ be proper, lsc and convex, i.e. $f \in \Gamma_0(H)$. Let $\bar{x} \in \text{int dom } f$. Then, there exists a continuous affine minorant φ of f such that $\varphi(\bar{x}) = f(\bar{x})$. In other words, $\exists u \in H$, such that*

$$(\forall x \in H) \quad \langle u, x - \bar{x} \rangle + f(\bar{x}) \leq f(x).$$

If a function is proper, lsc and weakly convex, i.e. $f \in \gamma_\rho(H)$ for some $\rho \geq 0$, the following corollary immediately holds.

Corollary 7. *Let $f \in \Gamma_\rho(H)$. Let $\bar{x} \in \text{int dom } f$. Then, there exists a concave quadratic minorant φ of f such that $\varphi(\bar{x}) = f(\bar{x})$. In other words, for $a \geq \rho$, $\exists u \in H$, such that*

$$(\forall x \in H) \quad -a\|x\|^2 + \langle u, x - \bar{x} \rangle + f(\bar{x}) + a\|\bar{x}\|^2 \leq f(x).$$

It is natural to define the following subdifferential.

Definition 8. Let $f : H \rightarrow (-\infty, +\infty]$ and $\bar{x} \in \text{dom } f$.

$$\partial_a^\Phi f(\bar{x}) := \{u \in H \mid (\forall x \in H) \quad f(x) - f(\bar{x}) \geq \varphi(x) - \varphi(\bar{x})\}, \quad (2.4)$$

where $\varphi \in \Phi_a$ and Φ_a , $a \geq 0$, is the class of elementary functions of the type

$$\Phi_a := \{\varphi : H \rightarrow \mathbb{R} \mid \varphi(x) := -a\|x\|^2 + \langle u, x \rangle + c, u \in H, c \in \mathbb{R}\}.$$

Subdifferentials of the type described in Definition 8, denoted Φ -subdifferentials, can be easily generalized by considering different classes of functions Φ and $\varphi \in \Phi$. The subdifferential ∂_a^Φ is a useful tool in Abstract convex analysis, see e.g. [Bednarczuk and Syga \(2022b\)](#); [Pallaschke and Rolewicz \(2013\)](#); [Rubinov \(2013\)](#).

Let f be proper, $\bar{x} \in \text{dom } f$ and $u \in \partial_a^\Phi f(\bar{x})$. Then,

$$f(x) - f(\bar{x}) \geq \langle u, x - \bar{x} \rangle - a\|x\|^2 + a\|\bar{x}\|^2 = \langle u - 2a\bar{x}, x - \bar{x} \rangle - a\|x - \bar{x}\|^2$$

We define the proximal subdifferential as follows.

Definition 9. Let $f : H \rightarrow (-\infty, +\infty]$ be proper and $\bar{x} \in \text{dom } f$. For $a \geq 0$, the proximal subdifferential is defined as follows.

$$\partial_a f(\bar{x}) := \{u \in H \mid f(x) - f(\bar{x}) \geq \langle u, x - \bar{x} \rangle - a\|x - \bar{x}\|^2 \ (\forall x \in H)\}. \quad (2.5)$$

The proximal subdifferential of a ρ -weakly convex function, defined in Definition 9, coincides with Clarke and Dini subdifferential when $a \geq \rho/2$, see [Jourani \(1996\)](#).

It is also possible to define ε -subdifferentials as follows.

Definition 10. Let $f : H \rightarrow (-\infty, +\infty]$ be proper, $\bar{x} \in \text{dom } f$ and $\varepsilon \geq 0$. We define the following ε -subdifferentials.

- Convex ε -subdifferential, defined as

$$\partial_0^\varepsilon f(\bar{x}) := \{u \in H \mid (\forall x \in H) \ f(x) - f(\bar{x}) \geq \langle u, x - \bar{x} \rangle - \varepsilon\}. \quad (2.6)$$

- The (Φ, ε) -subdifferential,

$$\partial_a^{\Phi, \varepsilon} f(\bar{x}) := \{u \in H \mid (\forall x \in H) \ f(x) - f(\bar{x}) \geq -a\|x\|^2 + \langle u, x - \bar{x} \rangle + a\|\bar{x}\|^2 - \varepsilon\}. \quad (2.7)$$

- The proximal ε -subdifferential

$$\partial_a^\varepsilon f(\bar{x}) := \{u \in H \mid (\forall x \in H) \ f(x) - f(\bar{x}) \geq \langle u, x - \bar{x} \rangle - a\|x - \bar{x}\|^2 - \varepsilon\}. \quad (2.8)$$

2.4 Outer approximation set for weakly convex constraints

Let $f_i \in \Gamma_{\rho_i}(H)$, $i = 1, \dots, m$, and consider the constraints $f_i(x) \leq 0$. Let $\bar{x} \in H$ and

$$I(\bar{x}) := \{i \in \{1, \dots, m\} \mid f_i(\bar{x}) > 0\}. \quad (2.9)$$

The central idea of this work is to generate an outer approximation set for

$$\mathcal{A} := \{x \in H \mid f_i(x) \leq 0, i = 1, \dots, m\} \quad (2.10)$$

with the help of the Φ -subdifferential, as described in the following.

For $u_i \in \partial_a^\Phi f_i(\bar{x})$, by (2.4), for $\varphi \in \Phi^a$, $\varphi(x) := -a_i \|x\|^2 + \langle u_i, x \rangle$,

$$(\forall x \in H, i \in I(\bar{x}) \quad f_i(x) \geq \varphi(x) - \varphi(\bar{x}) + f_i(\bar{x}). \quad (2.11)$$

We define the *quadratic cuts* as

$$(\forall x \in H, i \in I(\bar{x})) \quad L_i(\varphi, \bar{x})(x) := \varphi(x) - \varphi(\bar{x}) + f_i(\bar{x}), \quad (2.12)$$

where $\varphi \in \Phi^a$ defined in (2.4).

Proposition 11. Let $f_i \in \Gamma_{\rho_i}(H)$, $i = 1, \dots, m$, and consider the constraints $f_i(x) \leq 0$. Let $\bar{x} \in H$, $f_i(\bar{x}) > 0$ for $i \in I(\bar{x}) \subset \{1, \dots, m\}$. Let $u_i, \varphi, L_i(\varphi, \bar{x})(x)$ be defined as in (2.12). Then

$$\mathcal{A} \subset \{x \in H \mid L_i(\varphi, \bar{x})(x) \leq 0, \quad i \in I(\bar{x})\} =: \mathcal{O}(\bar{x}) \quad (2.13)$$

Proof. Let $x \in \mathcal{A}$, $f_i(x) \leq 0$, $i = 1, \dots, m$. By (2.12), $0 \geq f_i(x) \geq L_i(\varphi, \bar{x})(x)$. In conclusion, $\mathcal{A} \subset \mathcal{O}(\bar{x})$ for any $\bar{x}$ which is outside $\mathcal{A}$. $\square$

2.5 Calculus rules

In sections 2.5 and 2.6 we study more deeply the properties of the proximal subdifferential. These properties appear be useful in the convergence analysis of classical algorithms within the framework of weak convexity, see e.g. [Bednarczuk et al. \(2023c\)](#). In the literature, there exist numerous results providing calculus rules for the Fréchet, the limiting and the proximal subdifferentials, see *e.g.* [Jourani \(1996\)](#); [Kruger \(2003\)](#);

[Mordukhovich and Shao \(1995\)](#); [Mordukhovich and Nam \(2007\)](#); [Thibault \(1991\)](#) and many others. The main result of the present section is stated in [Theorem 15](#), where we provide the conditions for a sum rule for the global proximal ε -subdifferentials (in the sense of [Definition 10](#)) of the sum of two weakly convex functions. The proposed result allows us to extend the sum rule in ([Jourani, 1996](#), Theorem 5.1) – proved for exact proximal subdifferentials in normed spaces – to proximal ε -subdifferentials in Hilbert spaces: the interesting aspect of such rule is that it allows us to trace the modulus of proximal subdifferentiability.

The following notion of ρ -conjugate function will be used in the proof of [Theorem 15](#).

Definition 12. Let H be a Hilbert space. Let $f : H \rightarrow (-\infty, +\infty]$ be a proper function. For every $\rho \geq 0$ the function $(h)_\rho^* : H \rightarrow [-\infty, +\infty]$ defined as

$$(f)_\rho^*(u) := \sup_{y \in H} \left\{ -\frac{\rho}{2} \|y\|^2 + \langle u, y \rangle - f(y) \right\} \quad (2.14)$$

is called ρ -conjugate of f at $u \in H$ (when $\rho = 0$ we obtain the definition of the conjugate as defined in convex analysis and in this case we omit the subscript).

We recall the following classical result for the sum rule of convex subdifferentials.

Theorem 13 (([Rockafellar, 1966](#), Theorem3)). *Let H be a Hilbert space. Let $f_0, f_1 : H \rightarrow (-\infty, +\infty]$ be proper convex functions. Assume that $\text{dom } f_0 \cap \text{dom } f_1$ contains*

a point at which either f_0 or f_1 is continuous. Then, for all $s, x \in H$ we have

$$\begin{aligned} (a) \quad (f_0 + f_1)^*(s) &= \min_{\substack{p_0, p_1 \in H \\ s = p_0 + p_1}} \{f_0^*(p_0) + f_1^*(p_1)\} \\ (b) \quad \partial_0(f_0 + f_1)(x) &= \partial_0 f_0(x) + \partial_0 f_1(x) \end{aligned} \tag{2.15}$$

Remark 2. By (Bauschke and Combettes, 2017a, Theorem 15.3) (see also Attouch and Brezis (1986)) , point (a) of Theorem 13 can be proved under the assumption that f_0 and f_1 are convex proper lsc functions such that the conical hull of $\text{dom } f_0 - \text{dom } f_1$ is a closed linear subspace, i.e.,

$$0 \in \text{sri}(\text{dom } f_0 - \text{dom } f_1),$$

where sri denotes the strong relative interior, see (Bauschke and Combettes, 2017a, Def. 6.9). The regularity assumption in Attouch and Brezis (1986) is more general than the one in Rockafellar (1966) (see (Attouch and Brezis, 1986, Remark 1.3)). However, in some cases it is easier to verify the regularity condition requested in Rockafellar (1966).

The following proposition provides an auxiliary result used in the proof of Theorem 15.

Proposition 14. (Bednarczuk et al., 2023a, Proposition 3) Let H be a Hilbert space. For $i = 0, 1$, let function $f_i \in \Gamma_{\rho_i}(H)$ with $\rho_i \geq 0$. Assume that $\text{dom } f_0 \cap \text{dom } f_1$ contains a point at which either f_0 or f_1 is continuous. Then the following holds: for

any $s \in \text{dom}(f_0 + f_1)^*_\rho$, there exist $p_0, p_1 \in H$ such that $s = p_0 + p_1$ and

$$(f_0 + f_1)^*_{\rho_0 + \rho_1}(s) = (f_0)^*_{\rho_0}(p_0) + (f_1)^*_{\rho_1}(p_1). \quad (2.16)$$

Proof. We have that $\bar{f}_0(\cdot) = f_0(\cdot) + \rho_0/2 \|\cdot\|^2$ and $\bar{f}_1(\cdot) = f_1(\cdot) + \rho_1/2 \|\cdot\|^2$ are convex.

By Theorem 13, there exist p_0, p_1 such that $s = p_0 + p_1$ and

$$(\bar{f}_0 + \bar{f}_1)^*(s) = (\bar{f}_0)^*(p_0) + (\bar{f}_1)^*(p_1) \quad (2.17)$$

Notice that for $i = 0, 1$

$$\bar{f}_i^*(\cdot) := \sup_{y \in H} \{ \langle \cdot, y \rangle - \bar{f}_i(y) \} = \sup_{y \in H} \left\{ \langle \cdot, y \rangle - f_i(y) - \frac{\rho_i}{2} \|y\|^2 \right\} = (f_i)^*_{\rho_i}(\cdot) \quad (2.18)$$

so

$$(\bar{f}_0 + \bar{f}_1)^*(s) = (f_0)^*_{\rho_0}(p_0) + (f_1)^*_{\rho_1}(p_1). \quad (2.19)$$

In conclusion,

$$(f_0 + f_1)^*_{\rho_0 + \rho_1}(s) = (f_0)^*_{\rho_0}(p_0) + (f_1)^*_{\rho_1}(p_1). \quad (2.20)$$

□

We are ready to prove the following sum rule for proximal ε -subdifferentials $\partial_{\rho/2}^\varepsilon$. This result generalizes (Hiriart-Urruty and Lemaréchal, 2013, Theorem 3.1.1) and (Zalinescu, 2002, Theorem 2.8.7), which are formulated for convex functions and convex subdifferentials. An important aspect of our result is that it allows us to

trace the modulus of proximal subdifferentiability (as related to the modulus of weak convexity of the functions involved).

Theorem 15 (Sum Rule for ε -subdifferential). (*Bednarczuk et al., 2023a, Theorem 2*) *Let H be a Hilbert space. For $i = 0, 1$, let function $f_i \in \Gamma_{\rho_i}(H)$ with $\rho_i \geq 0$. Then, for all $x \in \text{dom } f_0 \cap \text{dom } f_1$ and for all $\varepsilon_0, \varepsilon_1 \geq 0$ we have*

$$\partial_{\rho_0/2}^{\varepsilon_0} f_0(x) + \partial_{\rho_1/2}^{\varepsilon_1} f_1(x) \subseteq \partial_{\rho/2}^{\varepsilon} (f_0 + f_1)(x) \quad (2.21)$$

for all $\varepsilon \geq \varepsilon_0 + \varepsilon_1$ and for all $\rho \geq \rho_0 + \rho_1$. The equality

$$\partial_{(\rho_0+\rho_1)/2}^{\varepsilon} (f_0 + f_1)(x) = \bigcup_{\{\varepsilon_0, \varepsilon_1 \mid \varepsilon_0 + \varepsilon_1 \leq \varepsilon\}} \partial_{\rho_0/2}^{\varepsilon_0} f_0(x) + \partial_{\rho_1/2}^{\varepsilon_1} f_1(x) \quad (2.22)$$

holds when $\text{dom } f_0 \cap \text{dom } f_1$ contains a point at which either $f_0 + \rho_0/2 \|\cdot\|^2$ or $f_1(\cdot) + \rho_1/2 \|\cdot\|^2$ is continuous.

Proof. For $x \in H$, if $w \in \partial_{\rho_0/2}^{\varepsilon_0} f_0(x)$, $v \in \partial_{\rho_1/2}^{\varepsilon_1} f_1(x)$, then it is clear that $w + v \in \partial_{(\rho_0+\rho_1)/2}^{\varepsilon} (f_0 + f_1)(x)$. Hence the inclusion (2.21) is satisfied.

To prove the equality in (2.22), let us consider $x \in \text{dom } f_0 \cap \text{dom } f_1$ and $u \in \partial_{\rho/2}^{\varepsilon} (f_0 + f_1)(x)$, where $\rho = \rho_0 + \rho_1$. By (Bednarczuk and Syga, 2022a, Theorem 2.4.ii, Eq. (5)) we have

$$(f_0 + f_1)(x) + (f_0 + f_1)^*_{\rho}(u + \rho x) \leq -\frac{\rho}{2} \|x\|^2 + \langle u + \rho x, x \rangle + \varepsilon \quad (2.23)$$

The inequality in (2.23) implies that $u + \rho x \in \text{dom}(f_0 + f_1)_\rho^*$. By applying Proposition 14, there exist two elements $p_0, p_1 \in H$ such that $u + \rho x = p_0 + p_1$ and

$$(f_0 + f_1)_\rho^*(u + \rho x) = f_{\rho_0}^*(p_0) + (f_1)_{\rho_1}^*(p_1) \quad (2.24)$$

so that (2.23) can be rewritten as

$$(f_0 + f_1)(x) + (f_0)_{\rho_0}^*(p_0) + (f_1)_{\rho_1}^*(p_1) \leq -\frac{\rho}{2}\|x\|^2 + \langle p_0 + p_1, x \rangle + \varepsilon \quad (2.25)$$

for all $x \in \text{dom } f_0 \cap \text{dom } f_1$. We now define the following values

$$\varepsilon_0 := f_0(x) + (f_0)_{\rho_0}^*(p_0) - \langle p_0, x \rangle + \frac{\rho_0}{2}\|x\|^2 \geq 0 \quad (2.26)$$

$$\varepsilon_1 := f_1(x) + (f_1)_{\rho_1}^*(p_1) - \langle p_1, x \rangle + \frac{\rho_1}{2}\|x\|^2 \geq 0 \quad (2.27)$$

which are positive in view of the definition of ρ -conjugate. Notice that (2.26) and (2.27) can be rewritten as

$$\varepsilon_0 = f_0(x) + (f_0)_{\rho_0}^*((p_0 - \rho_0 x) + \rho_0 x) - \langle (p_0 - \rho_0 x) + \rho_0 x, x \rangle + \frac{\rho_0}{2}\|x\|^2 \quad (2.28)$$

$$\varepsilon_1 = f_1(x) + (f_1)_{\rho_1}^*((p_1 - \rho_1 x) + \rho_1 x) - \langle (p_1 - \rho_1 x) + \rho_1 x, x \rangle + \frac{\rho_1}{2}\|x\|^2 \quad (2.29)$$

from which, by applying (Bednarczuk and Syga, 2022a, Theorem 2.4.ii, Eq. (5)) in a similar fashion as in (2.23), we obtain

$$(p_0 - \rho_0 x) \in \partial_{(2, \rho_0/2)}^{\varepsilon_0} f_0(x) \quad (2.30)$$

$$(p_1 - \rho_1 x) \in \partial_{(2, \rho_1/2)}^{\varepsilon_1} f_1(x) \quad (2.31)$$

which completes the proof. $\square$

In the following theorem we show that, in presence of differentiable functions, the notion of proximal ε -subdifferentials allows us to infer an inclusion which involves the gradient of the differentiable function.

Theorem 16. (*Bednarczuk et al., 2023a, Theorem 3*) *Let H be a Hilbert space. Let $f_0 : H \rightarrow (-\infty, +\infty]$ be proper, convex and differentiable with a L_0 -Lipschitz continuous gradient on the whole space H . Let $f_1 : H \rightarrow (-\infty, +\infty]$ be proper and ρ -weakly convex on H with $\rho \geq 0$. Then, we have the following inclusion*

$$\partial_{(2, \rho/2)}^{\varepsilon}(f_0 + f_1)(x) \subset \nabla(f_0)(x) + \partial_{(2, \rho/2 + L_0/2)}^{\varepsilon} f_1(x) \quad (2.32)$$

for all $x \in \text{dom } f_1$ such that $\partial_{(2, \rho/2)}^{\varepsilon} f_1(x) \neq \emptyset$.

Proof. In view of (Bauschke and Combettes, 2017a, Lemma 2.64), both f_0 and $-f_0$ are proximally L_0 -subdifferentiable on H . Precisely, for f_0 we have that for every $x, y \in H$

$$(-f_0)(y) - (-f_0)(x) \geq \langle \nabla(-f_0)(x) \mid y - x \rangle - \frac{L_0}{2} \|y - x\|^2. \quad (2.33)$$

Let us choose $x \in \text{dom } f_1$. If $\partial_{\rho/2}^{\varepsilon}(f_0 + f_1)(x) = \emptyset$, nothing needs to be proved.

Otherwise, we take $v \in \partial_{\rho/2}^\varepsilon(f_0 + f_1)(x)$. For every $y \in H$, by adding

$$(f_1 + f_0)(y) - (f_1 + f_0)(x) \geq \langle v \mid y - x \rangle - \frac{\rho}{2} \|y - x\|^2 - \varepsilon \quad (2.34)$$

and (2.33) we get

$$\partial_{(2,\rho/2)}^\varepsilon(f_1 + f_0)(x) + \nabla(-f_0)(x) \subset \partial_{(2,\rho/2+L_0/2)}^\varepsilon f_1(x) \quad (2.35)$$

i.e.

$$\partial_{(2,\rho/2)}^\varepsilon(f_1 + f_0)(x) \subset \partial_{(2,\rho/2+L_0/2)}^\varepsilon f_1(x) + \nabla(f_0)(x). \quad (2.36)$$

□

Remark 3. We highlight the following facts.

- For $\varepsilon > 0$, if f_1 is lower semicontinuous, the non-emptiness of the proximal subdifferential is ensured by (Bednarczuk et al., 2023a, Proposition 2).
- For $\varepsilon > 0$, Theorem 16 still holds when f_1 is convex, i.e. $\rho = 0$, but, in the right hand side of the inclusion (2.32), we have the proximal subdifferential with $a = L_0/2$, i.e.

$$\partial_0^\varepsilon(f_0 + f_1)(x) \subset \nabla(f_0)(x) + \partial_{(2,L_0/2)}^\varepsilon f_1(x).$$

2.6 Inexact proximal maps

In general settings, the computation of the proximal map needs to be addressed as an independent optimization problem. Some practical examples involves non-convex ℓ_p -seminorms (*i.e.* $p \in (0, 1)$) or the convex ℓ_p -norms (*i.e.* $p \geq 1$), unless p takes some specific values [Chaux et al. \(2007\)](#). Another example is given by the combination of a sparsity-promoting functions with a non-orthogonal linear operator, as in the case of the popular discrete Total Variation functional [Rudin et al. \(1992\)](#) (and its non-convex modifications), which has been extensively used in the context of image and signal processing. In these cases, at each point, the proximal map is defined up to a certain degree of accuracy and in the framework of proximal algorithms, it is important to carry out a convergence analysis that takes this fact into account. In order to do so, we consider the concept of ε -solution for an optimization problem and the related notion of ε -proximal point.

Definition 17 (ε -proximal point). Let H be a normed vector space. Let function $f : H \rightarrow (-\infty, +\infty]$ be proper and bounded from below and $\alpha > 0$. Then for all $y \in H$ and for all $\varepsilon \geq 0$, any ε -solution x_ε , $\varepsilon \geq 0$, to the proximal minimization problem

$$\underset{x \in H}{\text{minimize}} \quad f(x) + \frac{1}{2\alpha} \|x - y\|^2, \quad (2.37)$$

satisfies

$$(\forall x \in H) \quad f(x_\varepsilon) + \frac{1}{2\alpha} \|x_\varepsilon - y\|^2 \leq f(x) + \frac{1}{2\alpha} \|x - y\|^2 + \varepsilon,$$

and x_ε is said to be a ε -proximal point for f at y with respect to α . The set of all ε -proximal points of f at y with respect to α is denoted as

$$\varepsilon\text{-prox}_{\alpha f}(y) := \{x \in H \mid x \text{ is a } \varepsilon\text{-solution of (2.37)}\} \quad (2.38)$$

In the following result, we provide a relationship between the ε -proximal operator and the ε -proximal subdifferentials of weakly convex function, using the sum rule from Theo. 16. Specifically, in Prop. 18, we trace the constant C of the $(2, C)$ -subdifferential of f .

Proposition 18. (Bednarczuk et al., 2023a, Proposition 4) Let H be a Hilbert space. Let $f : H \rightarrow (-\infty, +\infty]$ be a proper, lower semicontinuous ρ -weakly convex function that is bounded from below on H . Let $\varepsilon \geq 0$, $\alpha > 0$. Then for every $y \in H$, $x_\varepsilon \in \varepsilon\text{-prox}_{\alpha f}(y)$ implies

$$\frac{y - x_\varepsilon}{\alpha} \in \partial_{(2, \rho/2 + 1/(2\alpha))}^\varepsilon f(x_\varepsilon). \quad (2.39)$$

Proof. By the definition of ε -proximal point and Fermat's optimality condition, (cf. Bauschke and Combettes, 2017b, Theorem 16.3) we have

$$x_\varepsilon \in \varepsilon\text{-prox}_{\alpha f}(y) \implies 0 \in \partial_{(2, \rho/2)}^\varepsilon \left(\frac{1}{2\alpha} \|\cdot - y\|^2 + f(\cdot) \right) (x_\varepsilon). \quad (2.40)$$

The assumptions in Theo. 16 are satisfied since $f_0(\cdot) = \frac{1}{2\alpha} \|\cdot - y\|^2$ is differentiable

on the whole space and its gradient has a Lipschitz constant $L_0 = 1/\alpha$, hence we also have the inclusion

$$0 \in \nabla \left(\frac{1}{2\alpha} \|\cdot - y\|^2 \right) (x_\varepsilon) + \partial_{(2,\rho/2+1/(2\alpha))}^\varepsilon f(x_\varepsilon) = \frac{x_\varepsilon - y}{\alpha} + \partial_{(2,\rho/2+1/(2\alpha))}^\varepsilon f(x_\varepsilon) \quad (2.41)$$

which is equivalent to

$$\frac{y - x_\varepsilon}{\alpha} \in \partial_{(2,\rho/2+1/(2\alpha))}^\varepsilon f(x_\varepsilon). \quad (2.42)$$

□

Remark 4. ([Bednarczuk et al., 2023a](#), Remark 6) We highlight that (2.39) is related to the notion of Type-2 approximation of the proximal point that is proposed in [Rasch and Chambolle \(2020\)](#); [Salzo and Villa \(2012\)](#) in the convex settings. In other words, by using the ε -proximal subdifferential instead of the (convex) ε -subdifferential, we can obtain a Type 2 approximation of the proximal point directly from Def. 17. This is due to Theo. 16, at the expense of increasing the modulus of proximal subdifferentiability by $1/(2\alpha)$.

Remark 5. ([Bednarczuk et al., 2023a](#), Remark 7)[ε -Subdifferential of a quadratic function] Let H be a Hilbert space. As a consequence of ([Hiriart-Urruty and Lemaréchal, 2013](#), Example 1.2.2), for a function of the form $f_0(x) = \frac{1}{2\alpha} \|x - y\|^2$ for some $y \in H$ and $\alpha > 0$, we have

$$\partial_0^\varepsilon f_0(x) = \left\{ \frac{x - y}{\alpha} + \frac{e}{\alpha} \mid \frac{1}{2\alpha} \|e\|^2 \leq \varepsilon \right\}. \quad (2.43)$$

In view of Remark 5 and Theo. 15, we can provide another interpretation for the ε -proximal points of a ρ -weakly convex function in terms of proximal ε -subdifferentials, where this time we are able to trace the modulus of weak convexity of the function.

Proposition 19. (Bednarczuk et al., 2023a, Proposition 5) Let H be a Hilbert space. Let $f : H \rightarrow (-\infty, +\infty]$ be a proper, lower semicontinuous ρ -weakly convex function that is bounded from below on H and let $\varepsilon \geq 0$, $\alpha > 0$. If $x_\varepsilon \in \varepsilon\text{-prox}_{\alpha f}(y)$, then there exist $\varepsilon_0, \varepsilon_1 \geq 0$ with $\varepsilon_0 + \varepsilon_1 \leq \varepsilon$ and there exists $e \in H$ with $\frac{\|e\|^2}{2\alpha} \leq \varepsilon_0$ such that

$$\frac{y - x_\varepsilon - e}{\alpha} \in \partial_{(2, \rho/2)}^{\varepsilon_1} f(x_\varepsilon). \quad (2.44)$$

If $\varepsilon = 0$ and $1/\alpha > \rho$, we obtain the equivalence.

Proof. By definition of ε -proximal point we have

$$x_\varepsilon \in \varepsilon\text{-prox}_{\alpha f}(y) \implies 0 \in \partial_{(2, \rho/2)}^\varepsilon \left(\frac{1}{2\alpha} \|\cdot - y\|^2 + f(\cdot) \right) (x_\varepsilon). \quad (2.45)$$

We can now apply Theo. 15, according to which there exist $\varepsilon_0, \varepsilon_1 \geq 0$ with $\varepsilon_0 + \varepsilon_1 \leq \varepsilon$ such that

$$0 \in \partial_0^{\varepsilon_0} \left(\frac{1}{2\alpha} \|\cdot - y\|^2 \right) (x_\varepsilon) + \partial_{(2, \rho/2)}^{\varepsilon_1} f(x_\varepsilon). \quad (2.46)$$

By applying Remark 5, we infer that there exists $e \in H$ with $\frac{\|e\|^2}{2\alpha} \leq \varepsilon_0$ such that

$$\frac{x_\varepsilon - y}{\alpha} + \frac{e}{\alpha} \in \partial_0^{\varepsilon_0} \left(\frac{1}{2\alpha} \|\cdot - y\|^2 \right) (x_\varepsilon) \quad (2.47)$$

which implies

$$\frac{y - x_\varepsilon - e}{\alpha} \in \partial_{(2,\rho/2)}^{\varepsilon_1} f(x_\varepsilon). \quad (2.48)$$

For the second part of the statement, we need to show that if $\frac{y-x_0}{\alpha} \in \partial_{(2,\rho/2)} f(x_0)$, then for every $x \in H$ the following inequality holds

$$f(x_0) + \frac{1}{2\alpha} \|x_0 - y\|^2 \leq f(x) + \frac{1}{2\alpha} \|x - y\|^2. \quad (2.49)$$

By definition of proximal subdifferential we have that for every $x \in H$

$$f(x) \geq f(x_0) + \left\langle \frac{y - x_0}{\alpha}, x - x_0 \right\rangle - \frac{\rho}{2} \|x - x_0\|^2 \quad (2.50)$$

Since $1/\alpha > \rho$ we have that for every $x \in H$

$$f(x) \geq f(x_0) + \left\langle \frac{y - x_0}{\alpha}, x - x_0 \right\rangle - \frac{1}{2\alpha} \|x - x_0\|^2 \quad (2.51)$$

We use the identity

$$\langle y - x_0, x - x_0 \rangle = -\frac{1}{2} \|y - x\|^2 + \frac{1}{2} \|x - x_0\|^2 + \frac{1}{2} \|y - x_0\|^2, \quad (2.52)$$

which follows from (Bauschke and Combettes, 2017a, Lemma 2.12). In conclusion, for every $x \in H$ we have

$$f(x_0) + \frac{1}{2\alpha} \|y - x_0\|^2 \leq f(x) + \frac{1}{2\alpha} \|y - x\|^2. \quad (2.53)$$

□

Remark 6. The interpretation provided by Proposition 19 is related to the notion of Type-1 approximation of the proximal point that is proposed in Rasch and Chambolle (2020); Salzo and Villa (2012) for convex functions.

Remark 7. (Bednarczuk et al., 2023a, Remark 8) Notice that (2.44) from Prop. 19 implies the inclusion (2.39) from Prop. 18, which is based on Theo. 16. By definition of proximal ε -subdifferentials, (2.44) is equivalent to

$$(x \in H) \quad f(x) - f(x_\varepsilon) \geq \left\langle \frac{y - x_\varepsilon}{\alpha}, x - x_\varepsilon \right\rangle - \frac{\rho}{2} \|x - x_\varepsilon\|^2 - \left\langle \frac{e}{\alpha}, x - x_\varepsilon \right\rangle - \varepsilon_1. \quad (2.54)$$

We consider the following estimation

$$\left\langle \frac{e}{\alpha}, x - x_\varepsilon \right\rangle \leq \varepsilon_0 + \frac{1}{2\alpha} \|x - x_\varepsilon\|^2 \quad (2.55)$$

which stems from Cauchy-Schwarz and Young's inequality. It follows that

$$f(x) - f(x_\varepsilon) \geq \left\langle \frac{y - x_\varepsilon}{\alpha}, x - x_\varepsilon \right\rangle - \left(\frac{\rho}{2} + \frac{1}{2\alpha} \right) \|x - x_\varepsilon\|^2 - \varepsilon \quad (2.56)$$

that is equivalent to (2.39)

$$\frac{y - x_\varepsilon}{\alpha} \in \partial_{(2, \rho/2+1/(2\alpha))}^\varepsilon f(x_\varepsilon). \quad (2.57)$$

The inclusion in (2.46) further leads to the following corollary which is a gen-

eralisation of ([Rasch and Chambolle, 2020](#), Lemma 2) from convex to proximal ε -subdifferentials.

Chapter 3

The cutting spheres method

3.1 Chapter Outline

- In Section 3.2, we show how to reformulate Problem (P0) as (P), in order to expand the class of problems which can be solved by the outer approximation algorithm described later in this chapter.
- In Section 3.3, we define the sequence of outer approximation sets produced by the Cutting spheres Algorithm 1.
- In Section 3.4, we describe the outer approximation algorithm, which we call cutting sphere algorithm 1.
- In Section 3.5, we provide a convergence proof for the cutting sphere algorithm 1. In particular, we prove that every convergent subsequence converge to a global minimum of (P).

3.2 Reformulating Problem (P0) as (P)

Recall that a function $f: \mathbb{R}^n \rightarrow]-\infty, +\infty]$ is ρ -weakly convex with $\rho \in \mathbb{R}_+$ if $f + \frac{\rho}{2}\|\cdot\|^2$ is convex.

The first step for solving problem (P0) is to rewrite it in the form of Problem (P).

Theorem 20. *Assume that Problem (1.1) has a solution. Let $(\rho, \eta) \in]0, +\infty[\times \mathbb{R}$ be such that*

$$D = \left\{ \hat{x} \in \text{Argmin}_{x \in S} F(x) \mid F(\hat{x}) + \eta \geq \frac{\rho}{2} \|\hat{x}\|^2 \right\} \neq \emptyset. \quad (3.1)$$

For every $i \in \{1, \dots, m\}$ and $y = (\bar{y}, y_n) \in \mathbb{R}^n$, let

$$\tilde{f}_i(y) = \begin{cases} F(\bar{y}) + \eta - \frac{\rho}{2} \|y\|^2 & \text{if } i = 1 \\ g_i(\bar{y}) & \text{otherwise.} \end{cases} \quad (3.2)$$

Then $\hat{x} \in D$ if and only if there exists $\hat{y}_n \in \mathbb{R}$ such that $\hat{y} = (\hat{x}, \hat{y}_n)$ is a solution to

$$\underset{y \in \mathbb{R}^n}{\text{minimize}} \quad \|y\|^2 \quad \text{s.t.} \quad (\forall i \in \{1, \dots, m\}) \tilde{f}_i(y) \leq 0. \quad (3.3)$$

Proof. By introducing a slack variable, Problem (1.1) is equivalent to

$$\underset{x \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}}{\text{minimize}} \quad x_n \quad \text{s.t.} \quad \begin{cases} F(x) + \eta \leq \frac{\rho}{2} x_n, \\ (\forall i \in \{2, \dots, m\}) g_i(x) \leq 0. \end{cases} \quad (3.4)$$

Since (3.1) holds, the feasible solutions (x, x_n) to (3.4), where $x \in D$, satisfy the

additional constraint

$$\|x\|^2 \leq x_n. \quad (3.5)$$

This shows that we can make the change of variables

$$\begin{aligned} x &= \bar{y} \\ x_n &= \|\bar{y}\|^2 + y_n^2 \\ y &= (\bar{y}, y_n), \end{aligned} \quad (3.6)$$

so leading to Problem (3.3). $\square$

Remark 8. Notice that (3.3) is equivalent to (P). If F and $(g_i)_{2 \leq i \leq m}$ are weakly convex (e.g., twice differentiable functions with a bounded Hessian), then the functions $(\tilde{g}_i)_{1 \leq i \leq m}$ are also weakly convex. However, if Problem (P0) is convex (i.e., $\beta_1 = \dots = \beta_m = 0$), Problem (3.3) has a weakly convex constraint, which is $g_0(y) \leq 0$.

Remark 9. If F is β_1 -weakly convex and, for every $i \in \{2, \dots, m\}$, g_i is β_i -weakly convex, then there exists convex functions H and $(h_i)_{1 \leq i \leq m}$ such that, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} F(x) &= H(x) - \frac{\beta_1}{2} \|x\|^2 \\ (\forall i \in \{2, \dots, m\}) \quad g_i(x) &= h_i(x) - \frac{\beta_i}{2} \|x\|^2. \end{aligned} \quad (3.7)$$

Therefore, Problem (P) can be rewritten as

$$\begin{aligned} & \underset{y=(\bar{y}, y_n) \in \mathbb{R}^n, \sigma \in [0, +\infty[}{\text{minimize}} && \sigma \quad \text{s.t.} && \begin{cases} H(\bar{y}) + \frac{\beta_1}{2} y_n^2 + \eta \leq \frac{\rho + \beta_1}{2} \sigma \\ (\forall i \in \{2, \dots, m\}) \quad h_i(\bar{y}) + \frac{\beta_i}{2} y_n^2 \leq \frac{\alpha_i}{2} \sigma \\ \sigma = \|y\|^2. \end{cases} \end{aligned} \quad (3.8)$$

We deduce that a convex relaxation of Problem (P) is given by

$$\begin{aligned} & \underset{y \in \mathbb{R}^n, \sigma \in [0, +\infty[}{\text{minimize}} && \sigma \quad \text{s.t.} && \begin{cases} H(\bar{y}) + \frac{\beta_1}{2} y_n^2 + \eta \leq \frac{\rho + \beta_1}{2} \sigma \\ (\forall i \in \{2, \dots, m\}) \quad h_i(\bar{y}) + \frac{\beta_i}{2} y_n^2 \leq \frac{\beta_i}{2} \sigma \\ \|y\|^2 \leq \sigma. \end{cases} \end{aligned} \quad (3.9)$$

If $(\hat{x}, \hat{y}_n, \hat{\sigma})$ is a solution to Problem (3.9), then $(\hat{x}, 0, \hat{\sigma})$ is a solution too. In turn, in the convex case, the pairs of the form $(\hat{x}, 0)$ are the solutions to Problem (3.4)-(3.5). (set $x_n = \sigma$). So, we can consider that Relaxation (3.9) becomes exact when F and $(g_i)_{2 \leq i \leq m}$ are convex.

3.3 Outer Approximation of set $\mathcal{A}$

In this section we define an outer approximation set $\mathcal{O}$ for the feasible set $\mathcal{A}$ of problem (P).

Since, for every $i \in I$, $f_i \in \Gamma_{\rho_i}(\mathbb{R}^n)$, by (7), for $\tilde{x} \in \text{dom } f_i$, if $\partial_{a_i}^\Phi f_i(\tilde{x}) \neq 0$, f_i has a quadratic tangent minorant of the form

$$L_i(\varphi, \tilde{x})(x) = \varphi(x) + c_i(\tilde{x}), \quad (3.10)$$

where $\varphi \in \Phi^a$ and $c_i(\tilde{x}) = f_i(\tilde{x}) + \varphi(\tilde{x})$. In Chapter 2, (2.12), we call $L_i(\varphi, \tilde{x})(x)$ quadratic cut.

Definition 21. Consider a sequence $(x_\ell)_{\ell \in \mathbb{N}}$ of $\mathbb{R}^n$. For every $\ell \in \mathbb{N}$, let

$$I(x_\ell) = \{i \in I \mid f_i(x_\ell) > 0\} \quad (3.11)$$

be the index set of violated constraints at x_ℓ . Let $k \in \mathbb{N}$, be such that, for every $\ell \in \{0, \dots, k\}$, $I(x_\ell) \neq \emptyset$ and, for every $i \in I(x_\ell)$, there exists $\varphi_i^\ell \in \partial_{a_i}^\Phi f_i(x_\ell)$. The *outer approximation set* $\mathcal{O}_k$ is defined as

$$\mathcal{O}_k := \{x \in \mathbb{R}^n \mid (\forall \ell \in \{0, \dots, k\})(\forall i \in I(x_\ell)) L_i(\varphi_i^\ell, x_\ell)(x) \leq 0\}, \quad (3.12)$$

where $L_i(\varphi_i^\ell, x_\ell)(x)$, $i \in I(x_\ell)$ are the quadratic cuts defined in (2.12).

Note that $\mathcal{O}_k$ with $k \geq 0$ defined above can be built recursively starting with $\mathcal{O}_{-1} = \mathbb{R}^n$ as

$$\mathcal{O}_k = \{x \in \mathbb{R}^n \mid (\forall i \in I(x_k)) L_i(\varphi_i^k, x_k)(x) \leq 0\} \cap \mathcal{O}_{k-1}. \quad (3.13)$$

The outer approximation terminology is justified by the following result.

Lemma 22. Consider the feasible set $\mathcal{A}$ of Problem (P) and a sequence $(x_k)_{k \in \mathbb{N}}$. Under the assumptions of Definition 21, the following properties are satisfied.

1. For every $i \in I(x_k)$, $f_i(x_k) = L_i(\varphi_i^k, x_k)(x_k) > 0$.
2. $\mathcal{A} \subset \mathcal{O}_k$.

Proof. 1. By (3.10),

$$L_i(\varphi_i^k, x_k)(x_k) = f_i(x_k) + \varphi_i^k(x_k) - \varphi_i^k(x_k) = f_i(x_k) > 0, \forall i \in I(x_k). \quad (3.14)$$

2. Take $x \in \mathcal{A}$, i.e. $f_i(x) \leq 0, \forall i \in I$. By (8), for all $k \in \mathbb{N}$, we have

$$0 \geq f_i(x) \geq \varphi_i^k(x) - \varphi_i^k(x_k) + f_i(x_k) = L_i(\varphi_i^k, x_k)(x), \text{ for all } i \in I(x_k) \quad (3.15)$$

where $\varphi_i^k \in \partial_{a_i}^\Phi f_i(x_k)$.

□

3.3.1 Geometric interpretation of the outer approximation constraints set

The following proposition links the quadratic cuts $L_i(\varphi_i^k, x_k)(x_k) \leq 0, i \in I, k \in \mathbb{N}$ and its geometrical properties.

Lemma 23. Given an index $j \in I$, $a_j > 0$ and an iteration number $k \in \mathbb{N}$, the constraint

$$L_j(\varphi_j^k, x_k)(x) \leq 0$$

describes the complement of a ball centered in $b_{jk}/2a_j$ with radius

$$r_k^2 = \|x_k - \frac{b_{jk}}{2a_j}\|^2 + \frac{f_j(x_k)}{a_j}. \quad (3.16)$$

Proof. The open ball $L_j(\varphi_j^k, x_k)(x) > 0$ is centered in $b_{jk}/2a_j$ by construction, hence

$L_j(\varphi_j^k, x_k)(x) = 0$ is equivalent to

$$\|x - \frac{b_{jk}}{2a_j}\|^2 = r_k^2, \quad (3.17)$$

for some $r_k \in \mathbb{R}_+$. We calculate the radius r_k .

$$L_j(\varphi_j^k, x_k)(x) = 0 \quad (3.18)$$

$$\varphi_j^k(x) - \varphi_j^k(x_k) + f_j(x_k) = 0 \quad (3.19)$$

$$-a_j\|x\|^2 + b_{jk}^T x + a_j\|x_k\|^2 - b_{jk}^T x_k + f_j(x_k) = 0 \quad (3.20)$$

$$-\|x - \frac{b_{jk}}{2a_j}\|^2 + \frac{\|b_{jk}\|^2}{4a_j^2} + \|x_k\|^2 - \frac{b_{jk}^T}{a_j} x_k + \frac{f_j(x_k)}{a_j} = 0 \quad (3.21)$$

$$\|x - \frac{b_{jk}}{2a_j}\|^2 = \frac{\|b_{jk}\|^2}{4a_j^2} + \|x_k\|^2 - \frac{b_{jk}^T}{a_j} x_k + \frac{f_j(x_k)}{a_j} \quad (3.22)$$

$$\|x - \frac{b_{jk}}{2a_j}\|^2 = \|x_k - \frac{b_{jk}}{2a_j}\|^2 + \frac{f_j(x_k)}{a_j}. \quad (3.23)$$

Hence, $r_k^2 = \|x_k - \frac{b_{jk}}{2a_j}\|^2 + \frac{f_j(x_k)}{a_j}$ (by construction, $f_j(x_k) > 0$ for all $k \in \mathbb{N}$). $\square$

3.3.2 2D Examples

The following 2D-examples shows how to build the outer approximation $\mathcal{O}$, given an initial point $x_0 \in H$.

Example 24. Let $f \in \Gamma_2(\mathbb{R}^2)$, $f(x) := ||x||^2 - 3| - 1$ and let $\mathcal{A} := \{x \in \mathbb{R}^2 \mid f(x) \leq 0\}$.

At $x_0 = (0, 0)$, $0 \in \partial_0(f + \|x\|^2)(x_0)$ and $\varphi := -\|x\|^2 \in \partial_2^\Phi f(x_0)$. Then, we can define the following outer approximation set for $\mathcal{A}$.

$$\mathcal{O} := \{x \in \mathbb{R}^2 \mid L(\varphi, x_0)(x) := -\|x\|^2 + 2 \leq 0\}.$$



Figure 3.1: The set $\mathcal{A}$ (left) in blue and the set $\mathcal{O}$ (right), in red, of Example 24. The picture was built with the 3D calculator of Geogebra, [Hohenwarter \(2002\)](#).

Example 25. Let $I := \{1, 2, 3\}$ and $f_i \in \Gamma_{\rho_i}(H)$, with $f_1(x) := x_1^2 + x_2^2 - 4 \leq 0$, $f_2(x) := -x_1^2 - x_2^2 + 1 \leq 0$ and $f_3(x) := -x^2 - (x_2 - 1)^2 + 0.5 \leq 0$. We define the set $\mathcal{A}_{ex2} := \{x \in H \mid f_1(x) \leq 0, f_2(x) \leq 0, f_3(x) \leq 0\}$. Let $x_0 = (0, 0)$. The set $\mathcal{A}_{ex2}$ is the same feasible set considered in ([Yang et al., 2018](#), Counterexample 2).

Notice that $I(x_0) = 2$. $f_2(x)$ is 2-weakly convex, $0 \in \partial_0(f_2 + \|\cdot\|^2)(x)$ and $\varphi_2(x) := -\|x\|^2$, $\varphi_2 \in \partial_2^\Phi f_2(x)$. We define the function $L_2(\varphi_2, x_0)(x) := \varphi_2(x) - \varphi_2(x_0) + f_2(x_0) = -\|x\|^2 + 1$. Hence, $\mathcal{O}_{ex2} := \{x \in \mathbb{R}^2 \mid -x_1^2 - x_2^2 + 1 \leq 0\}$.

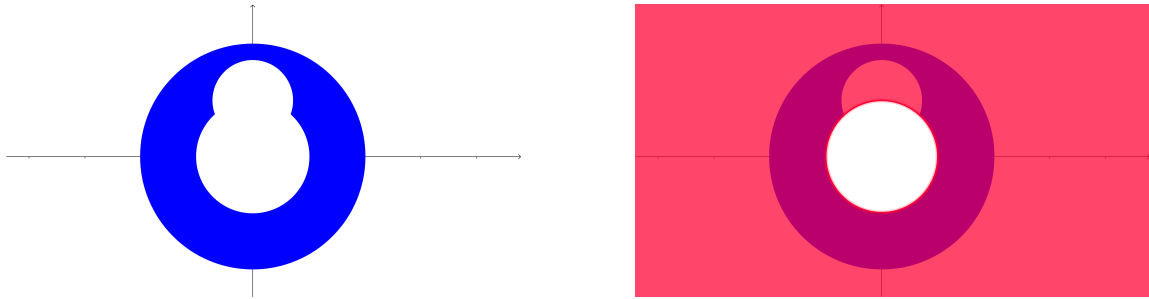


Figure 3.2: The set $\mathcal{A}_{ex2}$ (left) in blue and the set $\mathcal{O}_{ex2}$ (right) in red. The picture was built with the 3D calculator of Geogebra, [Hohenwarter \(2002\)](#).

3.4 Cutting spheres Algorithm

We recall that J is the objective of (P) and J^* is the value attained by J at the global minima of (P). We make the following standing assumption regarding the Cutting spheres Algorithm.

Assumption 26.

1. $\mathcal{A} \neq \emptyset$
2. There exists $\alpha > J^*$ such that the constraint functions $(f_i)_{i \in I}$ of (P) are continuous on an open set $C \supset E = \text{lev}_{\leq \alpha} J$.

Note that, since J is coercive and $\mathcal{A}$ is closed, Assumption 26.1 guarantees the existence of a solution to Problem (P).

Consider the following generic scheme of cutting sphere method.

Algorithm 1 Cutting spheres algorithm

Ensure: $x_0 = z$, $k = 0$, $\mathcal{O}_{-1} = \mathbb{R}^n$.**loop****if** $(\exists i \in I) f_i(x_k) > 0$ **then**Build $\mathcal{O}_k$ as in Definition 21.Solve $\min_{x \in \mathcal{O}_k} \|x - z\|^2$ and choose $x_{k+1} \in \arg \min_{x \in \mathcal{O}_k} \|x - z\|^2$ **else** $\mathcal{O}_k = \mathcal{O}_{k-1}$ $x_{k+1} = x_k$ **end if** $k \leftarrow k + 1$ **end loop**

Lemma 27. Under Assumption 26, Cutting spheres Algorithm 1 is well-defined and generates a sequence $(x_k)_{k \in \mathbb{N}}$ in E .

Proof. We will proceed by induction to show that, for every $k \in \mathbb{N}$, $x_k \in E$ and hence $\mathcal{O}_{k-1}$, defined in Definition 21, can be constructed. For $k = 0$, $x_k = z \in E$, since $J(z) = 0$. Let us show that $x_{k+1} \in E$.

If $(\forall i \in I) f_i(x_k) \leq 0$, we have that $\mathcal{O}_{k+1} = \mathcal{O}_k$ and $x_{k+1} = x_k \in E$. On the other hand, if the constraints are violated for indices in $I(x_k) \subset I$, to build $\mathcal{O}_k$, we need that $(\forall i \in I(x_k)) \partial_{a_i}^\Phi f_i(x_k) \neq \emptyset$. We know that $x_k \in E$ and $(f_i)_{i \in I(x_k)}$ are continuous on an open set which contains E , see Assumption (26). This implies that, for every $i \in I(x_k)$ and $a_i \geq \rho_i/2$, $\tilde{f}_i = f_i + a_i \|\cdot\|^2$ is a convex subdifferentiable function on E (see Definition 2), by (Bauschke and Combettes, 2017b, Proposition

16.17) and consequently $\partial_{a_i}^\Phi f_i(x_k) \neq \emptyset$. According to Lemma 22(2), $\emptyset \neq \mathcal{A} \subset \mathcal{O}_k$. As $x \mapsto \|x - z\|^2$ is a coercive continuous function, the existence of a minimizer x_{k+1} of this function is guaranteed on the closed nonempty set $\mathcal{O}_k$ and $J(x_{k+1}) \leq J^* < \alpha$, which shows that $x_{k+1} \in E$. $\square$

The outer approximation sets $\mathcal{O}_k$, $k \in \mathbb{N}$ introduced in Definition 21 produce a cumulative basis for Algorithm 1 in the sense of (Combettes, 2000, section 5.1), i.e. for every iteration $k \in \mathbb{N}$,

$$\mathcal{A} \subseteq \mathcal{O}_k \subseteq \mathcal{O}_{k-1} \subseteq \dots \subseteq \mathcal{O}_0. \quad (3.24)$$

We now establish the following necessary and sufficient condition for the Cutting Sphere Algorithm 1 to deliver a global optimal solution to (P) in a finite number of iterations (Theorem 30).

Lemma 28. Let $(x_k)_{k \in \mathbb{N}}$ be the sequence generated by Algorithm 1. Let $k \in \mathbb{N}$, and let $I(x_k)$ be the index set of violated constraints as defined by (3.11). Then, $I(x_k) = \emptyset$ if and only if $J^* = \min_{x \in \mathcal{O}_{k-1}} J(x)$.

Proof. Notice that

$$J(x_k) = \min_{x \in \mathcal{O}_{k-1}} J(x) \quad (3.25)$$

and, by Lemma 22(2), $\mathcal{A} \subset \mathcal{O}_{k-1}$. Hence, $J(x_k) \leq J^*$. We have then the following

equivalences:

$$\begin{aligned}
I(x_k) &= \emptyset \\
\Leftrightarrow x_k &\in \mathcal{A} \\
\Leftrightarrow J(x_k) &\geq J^* \\
\Leftrightarrow J(x_k) &= J^*, \tag{3.26}
\end{aligned}$$

where the last equivalence holds by (3.25). $\square$

3.5 Convergence proof of The Cutting Sphere Algorithm 1

The following preliminary result will be needed in our main convergence theorem, i.e. Theorem 30.

Theorem 29. *Suppose that Assumption 26 holds with $f_i \in \Gamma_{\rho_i}(\mathbb{R}^n)$, $i = 1, \dots, m$. Let $(x_\ell)_{\ell \in \mathbb{N}}$ be a sequence in $\mathbb{R}^n$ converging to $\bar{x} \in \mathbb{R}^n$. Let $i \in I$ and let $(b_i^\ell)_{\ell \in \mathbb{N}}$ be such that*

$$(\forall \ell \in \mathbb{N}) \quad b_i^\ell \in \partial_0 \tilde{f}_i(x_\ell),$$

where $\tilde{f}_i = f_i + a_i \|\cdot\|^2$ with $a_i \geq \rho_i/2$, and $\partial_0 \tilde{f}_i$ is its subdifferential in the sense of convex analysis (Definition 5). Let $\bar{x} \in \text{int dom } \tilde{f}_i$.

1. *There exists a convergent subsequence of $(b_i^\ell)_{\ell \in \mathbb{N}}$, and every convergent subsequence $(b_i^{\ell_k})_{k \in \mathbb{N}}$ converges to a subgradient of $\tilde{f}_i$ at $\bar{x}$.*
2. *If $\tilde{f}_i$ is differentiable at $\bar{x}$, then $b_i^\ell \rightarrow \nabla \tilde{f}_i(\bar{x})$.*

Proof.

1. The function $\tilde{f}_i$ is continuous in $\text{int dom } \tilde{f}_i$, see (Bauschke and Combettes, 2017b, Corollary 8.39). We deduce from (Bauschke and Combettes, 2017b, Proposition 16.17(iii)), that there exists a closed ball $B(\bar{x}, \rho)$ with center $\bar{x}$ and radius $\rho > 0$ such that $\partial_0 \tilde{f}_i(B(\bar{x}, \rho))$ is bounded. As there exists $\ell_0 \in \mathbb{N}$ such that $\{x_\ell\}_{\ell \geq \ell_0} \subset B(\bar{x}, \rho)$, $(b_i^\ell)_{\ell \in \mathbb{N}}$ is a bounded sequence, hence it has a convergent subsequence (Bauschke and Combettes, 2017b, Lemma 2.45).

Let $(b_i^{\ell_k})_{k \in \mathbb{N}}$ be a subsequence of $(b_i^\ell)_{\ell \in \mathbb{N}}$ converging to $\bar{b}_i$. It follows from (Bauschke and Combettes, 2017b, Proposition 16.36) that $\bar{b}_i \in \partial_0 \tilde{f}_i(\bar{x})$.

2. Assume that $\tilde{f}_i$ is differentiable at $\bar{x}$. Then $\partial_0 \tilde{f}_i(\bar{x}) = \{\nabla \tilde{f}_i(\bar{x})\}$. Since $(b_i^\ell)_{\ell \in \mathbb{N}}$ is bounded and every convergent subsequence has the same limit point, it converges to $\nabla \tilde{f}_i(\bar{x})$.

□

Remark 10. In view of the theorem used in the proof, the assertion of Theorem 29 is true in any Hilbert space.

In order to prove the convergence of Cutting spheres Algorithm 1, we will need the following observation.

Remark 11. For every $i \in I$, we choose a_i be a given value in $[\rho_i/2, +\infty[$, at every iteration $k \in \mathbb{N}$ of Cutting spheres Algorithm 1 such that $I(x_k) \neq \emptyset$, the subgradients $(\varphi_i^k)_{i \in I(x_k)}$ belong to the set

$$\{\varphi: \mathbb{R}^n \rightarrow \mathbb{R} \mid (\forall x \in \mathbb{R}^n) \varphi(x) = -a_i \|x\|^2 + \langle b, x \rangle + \tilde{c}, b \in \mathbb{R}^n, \tilde{c} \in \mathbb{R}\}, \quad (3.27)$$

It means that the same curvature constant a_i remain unchanged through the whole iteration process for each constraint function f_i when computing its sub-gradient. Now we formulate our main convergence result for the Cutting spheres Algorithm 1.

Theorem 30. *Consider Problem (P). Suppose that Assumption 26 holds. Let $(x_k)_{k \in \mathbb{N}}$ be the sequence generated by Algorithm 1. Then,*

1. $(x_k)_{k \in \mathbb{N}}$ possesses a convergent subsequence, and every sequential cluster point of $(x_k)_{k \in \mathbb{N}}$ is a global solution to (P);
2. if (P) has a unique solution, then $(x_k)_{k \in \mathbb{N}}$ converges to this solution;
3. $(J(x_k))_{k \in \mathbb{N}}$ is a non decreasing sequence converging to the optimal value J^* of (P).

Proof.

1. According to Lemma 27, $\{x_k\}_{k \in \mathbb{N}} \subset E$. Moreover, E is a nonempty closed ball in $\mathbb{R}^n$, hence it is compact. We deduce that $(x_k)_{k \in \mathbb{N}}$ possesses a subsequence converging to a point in E and that every sequential cluster point of $(x_k)_{k \in \mathbb{N}}$ belongs to E .

Now, we show that every converging subsequence of $(x_k)_{k \in \mathbb{N}}$ is a minimizing sequence, i.e., its limit $\bar{x}$ is such that $J(\bar{x}) = J^*$. We first note that, if at some iteration k , the obtained iterate x_k is feasible then, by Lemma 28, $x_k = \bar{x}$ is a global minimum of Problem (P) which means that the algorithm converges in

a finite number of steps. We can thus focus on the case when, for every $k \in \mathbb{N}$, x_k is infeasible for (P).

Consider a convergent subsequence $(x_{k_\ell})_{k_\ell \in \mathbb{N}}$ with limit point $\bar{x} \in E$. Let us consider a fixed $j \in I$. Two situations may arise:

- (a) There exists $\ell_0 \in \mathbb{N}$ such that, for every $\ell \geq \ell_0$, the j -th constraint is satisfied by x_{k_ℓ} , i.e., $f_j(x_{k_\ell}) \leq 0$. Then, by the continuity of f_j on E , $f_j(\bar{x}) \leq 0$.
- (b) There exists a subsequence $(x_{p_\ell})_{\ell \in \mathbb{N}}$ of $(x_{k_\ell})_{k_\ell \in \mathbb{N}}$ for which the j -th constraint is violated, i.e., for every $\ell \in \mathbb{N}$, $f_j(x_{p_\ell}) > 0$.

Let us investigate Case (1b). By construction, for every $\ell \in \mathbb{N}$ and $p > p_\ell$, $L_j(\varphi_j^{p_\ell}, x_{p_\ell})(x_p) \leq 0$ where $\varphi_j^{p_\ell} \in \partial_{a_j}^\Phi f_j(x_{p_\ell})$.

Since $L_j(\varphi_j^{p_\ell}, x_{p_\ell})(\cdot)$ is continuous, we deduce that

$$L_j(\varphi_j^{p_\ell}, x_{p_\ell})(\bar{x}) \leq 0. \quad (3.28)$$

According to Remark 11, $\varphi_j^{p_\ell}$ has the following form:

$$(\forall x \in \mathbb{R}^n) \quad \varphi_j^{p_\ell}(x) = -a_j \|x\|^2 + \langle b_j^{p_\ell}, x \rangle,$$

where $b_j^{p_\ell} \in \partial_0 \tilde{f}_j(x_{p_\ell})$ with $\tilde{f}_j = f_j + a_j \|\cdot\|^2$. Since $x_{p_\ell} \rightarrow \bar{x} \in E \subset \text{int dom } \tilde{f}_j$ (see (Bauschke and Combettes, 2017b, Corollary 8.39)), it follows from Theorem 29(1) that there exists a subsequence $(b_j^{q_\ell})_{\ell \in \mathbb{N}}$ of $(b_j^{p_\ell})_{\ell \in \mathbb{N}}$ converging to

$\bar{b}_j \in \partial_0 \tilde{f}_j(\bar{x})$. Using (3.28) and (3.10) yields

$$L_j(\varphi_j^{q_\ell}, x_{q_\ell})(\bar{x}) = f_j(x_{q_\ell}) + \varphi_j^{q_\ell}(\bar{x}) - \varphi_j^{q_\ell}(x_{q_\ell}) \leq 0. \quad (3.29)$$

Let us look at the asymptotic behavior of these terms as $\ell \rightarrow \infty$. By continuity,

$$f_j(x_{q_\ell}) \rightarrow f_j(\bar{x}) \geq 0. \quad (3.30)$$

Let us show that $\varphi_j^{q_\ell}(\bar{x}) - \varphi_j^{q_\ell}(x_{q_\ell}) \rightarrow 0$ as $\ell \rightarrow \infty$. By defining

$$(\forall x \in \mathbb{R}^n) \quad \bar{\varphi}_j(x) = -a_j \|x\|^2 + \langle \bar{b}_j, x \rangle,$$

we obtain the following inequality

$$\begin{aligned} |\varphi_j^{q_\ell}(\bar{x}) - \varphi_j^{q_\ell}(x_{q_\ell})| &\leq |\varphi_j^{q_\ell}(\bar{x}) - \bar{\varphi}_j(\bar{x})| + |\bar{\varphi}_j(\bar{x}) - \bar{\varphi}_j(x_{q_\ell})| + |\bar{\varphi}_j(x_{q_\ell}) - \varphi_j^{q_\ell}(x_{q_\ell})| \\ &= |\langle b_j^{q_\ell} - \bar{b}_j, \bar{x} \rangle| + |\bar{\varphi}_j(\bar{x}) - \bar{\varphi}_j(x_{q_\ell})| + |\langle \bar{b} - b_j^{q_\ell}, x_{q_\ell} \rangle|. \end{aligned} \quad (3.31)$$

The first and last terms converge to 0, because of the convergence of $(b_j^{q_\ell})_{\ell \in \mathbb{N}}$ to $\bar{b}_j$ and the boundedness of $(x_{q_\ell})_{\ell \in \mathbb{N}}$ (see Theorem 29). The second term converges to 0 by continuity of $\bar{\varphi}_j$. In view of this fact, we deduce from (3.29) and (3.30) that $f_j(\bar{x}) = 0$, which proves that $\bar{x}$ satisfies the j -th constraint.

By repeating the same reasoning for each constraint $j \in I$, we show that $\bar{x} \in \mathcal{A}$, hence $J(\bar{x}) \geq J^*$. According to Lemma 22(2) and the continuity of J ,

$$J(x_{k_\ell}) \leq J^* \Rightarrow J(\bar{x}) \leq J^*.$$

Then, $J^* = J(\bar{x})$ which completes the proof of 1.

2. If (P) has a unique solution, then $(x_k)_{k \in \mathbb{N}}$ is a bounded sequence with a unique sequential cluster point. Thus, it converges to this point.
3. By (3.24), $(J(x_k))_{k \in \mathbb{N}}$ is an increasing sequence which is upper bounded by J^* . In addition, it follows from 1. that $(J(x_k))_{k \in \mathbb{N}}$ has a subsequence converging to J^* . The limit of $(J(x_k))_{k \in \mathbb{N}}$ is thus equal to J^* .

□

Finding a global solution of the outer approximation problem, defined in (OP_k), is the scope of the next chapters. Still, we can show the behaviour of the algorithm in the simple case described by Example 25.

Example 31. We apply Algorithm 1 to

$$\min_{x \in \mathbb{R}^2} \|x\|^2 \text{ s.t. } x \in \mathcal{A}_{ex2}. \quad (3.32)$$

We start from $x_0 = (0, 0)$ and we build $\mathcal{O}_{ex2}$. If we choose $x_1 = (1, 0)$, which is an optimal solution of both Problem 3.32 and

$$\min_{x \in \mathbb{R}^2} \|x\|^2 \text{ s.t. } x \in \mathcal{O}_{ex2}, \quad (3.33)$$

the algorithm stops, since $I(x_1) = \emptyset$. On the other hand, we can set $x_1 = (0, 1)$, which is also a global solution of Problem (3.33), but not a solution of Problem 3.32. In this case $I(x_1) = \{3\}$. We restart the loop. We solve a new outer approximation

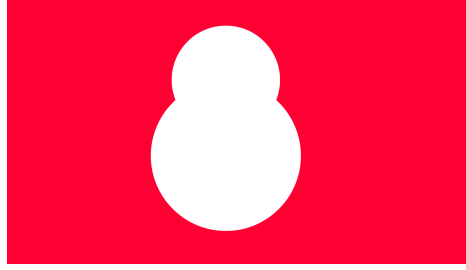


Figure 3.3: $\mathcal{O}_1$ in Example 31, colored in red. The picture was built with the 3D calculator of Geogebra, [Hohenwarter \(2002\)](#).

problem, which is

$$\min_{x \in \mathbb{R}^2} \|x\|^2 \text{ s.t. } x \in \mathcal{O}_1 := \{x \in \mathbb{R}^2 \mid L_3(\varphi_3^1, x_1)(x) \leq 0\} \cap \mathcal{O}_{ex2}, \quad (3.34)$$

and $L_3(\varphi_3^1, x_1)(x) := -\|x\|^2 + (0, 2)x - 0.5$. Since the solution set of Problem 3.34 is equal to the solution set of Problem 3.32, see Figure 3.3, the algorithm terminates when we choose x_2 as a global minimum of Problem 3.34.

Chapter 4

The Outer Approximation problem is a QCQP

4.1 Chapter Outline

- In Section 4.2, we show how the outer approximation problem belongs to the class of (nonconvex) QCQP problems and, more specifically, to the subclass of Scalar-Quadratically Constrained Quadratic Programs (S-QCQP) problems.
- In Section 4.3, we present some well known results on QCQP problems in $\mathbb{R}^n$. The aim of the following sections is to present the most important results in the literature about QCQP, which is an intensively discussed topic, and highlight the relations between these results.
- In Section 4.4, we recall the concepts and the relations between Lagrangian duality and convex relaxations of QCQP problems.
- In Section 4.5, we recall the concept of hidden convex QCQP and we investigate the relations between hidden convexity, convex relaxations and Lagrangian duality.

- In Section 4.6, we focus on QCQP problems in Hilbert space. We show that the convexity of the Generalized Image Set (GIS) is a sufficient condition for the S-Lemma to hold. We prove that, when the S-Lemma holds for a given QCQP problem, the KKT conditions are necessary and sufficient for global optimality.
- In Section 4.7, we consider S-QCQP in Hilbert spaces. By applying the S-Lemma, we provide conditions under which the KKT conditions are necessary and sufficient for global optimality.

4.2 From the Outer Approximation problem to S-QCQP

The outer approximation problem for (P) which appears at each step of the Cutting sphere Algorithm 1, presented in Definition 21, formula (3.12), takes the following form

$$\text{Min}_{x \in \mathbb{R}^n} \frac{1}{2} \|x - z\|^2 \quad x \in \mathcal{O}_k, \quad (OP_k)$$

where

$$\mathcal{O}_k := \{x \in \mathbb{R}^n \mid -a_i \|x\|^2 + \langle b_i, x \rangle + c_i \leq 0 \quad i \in \{1, \dots, m_k\}\}. \quad (4.1)$$

There always exists an optimal solution x^* for (OP_k) , since $z = \arg \min_{x \in \mathbb{R}^n} \frac{1}{2} \|x - z\|^2$ is the global minimum in the unconstrained case.

Problem (OP_k) belongs to the class of S-QCQP problems studied in Bednarczuk

and Brucola (2023), defined as

$$\begin{aligned} \min_{x \in H} q_0 &:= a_0 \|x\|^2 + \langle b_0, x \rangle + c_0 \\ q_i &:= a_i \|x\|^2 + \langle b_i, x \rangle + c_i \leq 0 \quad i \in \{1, \dots, m\} \end{aligned} \quad (\text{S-QCQP})$$

$a_i, c_i \in \mathbb{R}, b_i \in H, i \in \{0, \dots, m\}$.

4.3 QCQP problems in $\mathbb{R}^n$

4.3.1 Definitions and preliminaries

Consider a possibly nonconvex QCQP problem; which can be defined as follows in a real Hilbert space. A QCQP in Hilbert is defined as

$$\begin{aligned} \text{Min}_{x \in H} \quad & f_0(x) := \langle x, A_0 x \rangle + 2\langle b_0, x \rangle + c_0 \\ \text{s.t.} \quad & f_i(x) := \langle x, A_i x \rangle + 2\langle b_i, x \rangle + c_i \leq 0, \quad i = 1, \dots, m \end{aligned} \quad (\text{QCQP})$$

where the continuous linear self-adjoint operators $A_J, A_k : H \rightarrow H$, $b_J, b_k \in H$, and $c_J, c_k \in \mathbb{R}$ are given data for $k = 1, \dots, m$. The optimal value of QCQP is denoted by J^* .

Since the results of this section holds only in $\mathbb{R}^n$, we rewrite QCQP as

$$\begin{aligned} \text{Min}_{x \in \mathbb{R}^n} \quad & f_0(x) := x^T A_0 x + 2b_0^T x + c_0 \\ \text{s.t.} \quad & f_i(x) := x^T A_i x + 2b_i^T x + c_i \leq 0, \quad i = 1, \dots, m, \end{aligned} \quad (\text{QCQP})$$

where $A_i \in S^n$, S^n denotes the space of symmetric $(n \times n)$ matrices, $b_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$ are given data for $i = 0, \dots, m$.

The most frequently adopted relaxation for QCQP problems is the SDP (Semidefinite programming) relaxation, also called Shor relaxation [Shor \(1987\)](#).

Let $x \in \mathbb{R}^n$, let $X \in S^n$, which is the class of symmetric matrices in $\mathbb{R}^n$.

Remark 12. The scalar product between two matrices $A \in S^n$ and $X \in S^n$ is the trace inner product, defined as $Tr(AX) = \sum_{i=1}^n \sum_{j=1}^n A_{ij}X_{ij}$. The corresponding Froebenius norm is $\|A\| = \sqrt{Tr(AA)} = \sqrt{\sum_{i=1}^n \sum_{j=1}^n A_{ij}^2}$.

The SDP relaxation of Problem [\(QCQP\)](#) is the following.

$$\begin{aligned}
 & \text{Min}_{x \in \mathbb{R}^n} Tr(X, A_0) + 2b_0^T x + c_0 \\
 & \text{s.t. } Tr(X, A_i) + 2b_i^T x + c_i \leq 0, \quad i = 1, \dots, m \\
 & \text{s.t. } \begin{pmatrix} 1 & x^T \\ X & xx^T \end{pmatrix} \succeq 0
 \end{aligned} \tag{SDP}$$

We say that the relaxation [\(SDP\)](#) is *exact* if and only if the optimal value of [\(SDP\)](#) is equal to the optimal value of [\(QCQP\)](#) and we write $Opt(\text{QCQP}) = Opt(\text{SDP})$.

We say that the KKT conditions hold for [\(QCQP\)](#) at point x^* when there exists a vector $\gamma^* := (\gamma_1^*, \dots, \gamma_m^*) \in \mathbb{R}_+^m \setminus \mathbf{0}_m$ such that

$$\begin{aligned}
 (i) \quad & \nabla(f_0 + \sum_{k=1}^m \gamma_k^* f_k)(x^*) = 0 \\
 (ii) \quad & \gamma_k^* f_k(x^*) = 0 \quad k \in \{1, \dots, m\} \\
 (iii) \quad & A_0 + \sum_{k=1}^m \gamma_k^* A_k \succeq 0
 \end{aligned} \tag{KKT}$$

We define the cones S_+^n and S_{++}^n of symmetric matrices which are also positive

semidefinite and positive definite, respectively. Note that a set C is a cone if $C = \mathbb{R}_+ C$. Let C be a cone in the finite dimensional vector space $\mathcal{Y}$ equipped with the scalar product “ $\cdot$ ”. The polar cone of C is defined as follows:

$$C^- := \{y \in \mathcal{Y} : (\forall w \in C) \ y \cdot w \leq 0\} \quad (4.2)$$

Definition 32. Let $A, B \in S^n$. We define the Loewner Ordering, [Mathias \(1991\)](#), as

$$A \succeq B \text{ iff } A - B \in S_+^n \quad (4.3)$$

If a matrix A belongs to S_+^n then we write $A \succeq 0$; if $A \in S_{++}^n$ then $A \succ 0$. The interior of S_+^n is S_{++}^n , [Wang et al. \(2010\)](#).

Consider the quadratic functional $f_i(x) := x^T A_i x + 2b_i^T x + c_i$, where $A_i \in S^n$, $b_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$; $i = 0$ is the index associated with the objective function of (QCQP) and $i \in \{1, \dots, m\}$ are the indices which refer to the constraints.

Let $\det(A)$ denotes the determinant of a matrix A and $A^\dagger$ its pseudo-inverse. $\text{diag}(A) \in \mathbb{R}^n$ is the vector of the elements in the main diagonal of $A \in S^n$.

4.3.2 The SDP relaxation

The most frequently used relaxation for the QCQP problem is the SDP (Semidefinite programming) relaxation, also called Shor relaxation [Shor \(1987\)](#).

The aim of this section is to introduce the most important results related to the SDP relaxation of a general (QCQP).

Let $x \in \mathbb{R}^n$ and let $X \in S^n$. Problem (QCQP) can be equivalently rewritten as

follows (Vandenberghe and Boyd (1996), problem 19):

$$\begin{aligned}
 & \text{Min}_{x \in \mathbb{R}^n} \text{Tr}(XA_0) + 2b_0^T x + c_0 \\
 & \text{s.t. } \text{Tr}(XA_i) + 2b_i^T x + c_i \leq 0, \quad i = 1, \dots, m \\
 & \text{s.t. } X = xx^T
 \end{aligned} \tag{4.4}$$

Shor (1987), proposed to relax the latter constraint of (4.4). Let “ $\succeq$ ” define the Loewner Ordering as is Definition 32. Note that $X \succeq xx^T$ is equivalent to the following:

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \tag{4.5}$$

In fact the Schur complement $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$ with respect of 1 is $U_x := X - xx^T$ and by Boyd et al. (2004), section A5.5 we have $U_x \succeq 0$ if and only if (4.5) holds.

By relaxing the constraint $X = xx^T$ with (4.5), we obtain the relaxation (SDP) of (QCQP).

We also notice that the relaxation (SDP) of Problem (QCQP) is exact if and only if the smallest rank among the optimal solutions $R^* := \begin{pmatrix} 1 & (x^*)^T \\ x^* & X^* \end{pmatrix}$ of (SDP) satisfies $r^* = 1$, Vandenberghe and Boyd (1996).

The following theorem highlights an important property of convex (QCQP).

Theorem 33. (Park and Boyd (2017), Lemma 2) *In the case of a convex (QCQP), its (SDP) relaxation is exact.*

Proof. The proof pursues the following points:

1. Let $x \in \mathbb{R}^n$ and $X \in S^n$ with $X \succeq x$. The following inequality holds:

$$\text{Tr}(XA_i) + 2b_i^T x + c_i - (x^T A_i x + 2b_i^T x + c_i) = \text{Tr}((X - xx^T)A_i) \geq 0 \quad (4.6)$$

2. Let (x, X) be feasible for (SDP). Then x is feasible for (QCQP). In fact for any convex i -constraint of (QCQP), we have $A_i(x) \succeq 0$. Hence, by (4.6):

$$x^T A_i x + 2b_i^T x + c_i \leq \text{Tr}(XA_i) + 2b_i^T x + c_i \leq 0 \quad (4.7)$$

Since the feasible sets of (SDP) and (QCQP) are assumed to be convex, (4.7) implies the feasibility of x for (QCQP).

3. Consider a feasible and optimal point (x^*, X^*) for (SDP). Taking into account the convexity of the objective function, we have $A_0 \succeq 0$ and $x^{*T} A_0 x^* + 2b_0^T x^* + c_0 \leq \text{Tr}(X^* A_0) + 2b_0^T x^* + c_0$ by (4.6).

□

4.4 Lagrangian Duality for QCQP in $\mathbb{R}^n$

The Lagrangian duality is essential in the study of the relationship between (QCQP) and its (SDP) relaxation. Given a primal problem (P) with optimal value Opt(P) and a dual problem (D) with optimal value Opt(D) , the zero duality gap holds iff $\text{Opt(P)} = \text{Opt(D)}$. The strong duality holds when $\text{Opt(P)} = \text{Opt(D)}$ and moreover the optimum in the dual is attained. In this section we will prove that:

1. The Lagrangian Dual Problems of (QCQP) and (SDP) are the same.
2. The zero duality gap and the strong duality between (SDP) and the Lagrangian Dual require some mild assumptions, that we will state in Theorem 43 and Theorem 44.

The results presented in this section are not an original contribution of this work and can be find in e.g. Vandenberghe and Boyd (1996), where, however, the proofs of these results are omitted. We divide the rest of this section in two subsections. The first one is dedicated to formulate the Lagrangian Dual of (QCQP). The second subsection proves the strong duality between (SDP) and its Lagrangian Dual.

4.4.1 QCQP duality

Theorem 34. *For any $\gamma \in \mathbb{R}_+^m$, let $A(\gamma) := A_0 + \sum_{i=1}^m \gamma_i A_i$, $b(\gamma) := b_0 + \sum_{i=1}^m \gamma_i b_i$, $c(\gamma) := c_0 + \sum_{i=1}^m \gamma_i c_i$. Let $A(\gamma) \succeq 0$. The Lagrangian Dual of (QCQP) can be written as:*

$$\begin{aligned} & \text{Max}_{\gamma \geq 0, \tau} \quad \tau \\ & \text{s.t.} \quad Z(\gamma, \tau) := \begin{pmatrix} c(\gamma) - \tau & b(\gamma)^T \\ b(\gamma) & A(\gamma) \end{pmatrix} \succeq 0 \end{aligned} \tag{Dual_1}$$

Proof. The Lagrangian dual of (QCQP) is:

$$\text{Max}_{\gamma \geq 0} \Phi(\gamma) := \text{Min}_{x \in \mathbb{R}^n} \quad x^T A(\gamma) x + 2b(\gamma)^T x + c(\gamma) \tag{4.8}$$

Consider now the following dual objective, which is more general than $\Phi(\gamma)$ in

(4.8):

$$\Phi_*(x, v) := \min_{x \in \mathbb{R}^n, v \in \mathbb{R}^p} x^T A_* x + 2v^T B_*^T x + v^T C_* v \quad (4.9)$$

for $A_* \in S^n$, $B_* \in \mathbb{R}^{n \times p}$ and $C_* \in S^p$ and let $A_* \succeq 0$. We have:

$$\begin{aligned} \nabla_x(\Phi_*) &= 0 \\ A_* x &= -B_* v \end{aligned} \quad (4.10)$$

Equation (4.10) is solvable if and only if $B_* v \in \mathcal{R}(A_*)$, where $\mathcal{R}(A_*)$ is the range of A_* and $\mathcal{R}(A_*) := \{A_* x \mid x \in \mathbb{R}^n\}$.

Hence, if $A_* \succeq 0$ and $B_* v \in \mathcal{R}(A_*)$, the solution of (4.10) is $x = -A_*^\dagger B_* v$ and

$$\Phi_*(x, v) = v^T (C_* - B_*^T A_*^\dagger B_*) v \quad (4.11)$$

where $A^\dagger$ is the pseudo-inverse of A . If $A_* \not\succeq 0$ or $B_* v \notin \mathcal{R}(A)$, the minimization of $\Phi_*(x, v)$ over the variable x is unbounded. (Boyd et al. (2004), section A.5.5).

The matrix $C_* - B_*^T A_*^\dagger B_*$ can be seen as a generalization of the Schur complement of the matrix $\begin{pmatrix} C_* & B_*^T \\ B_* & A_* \end{pmatrix}$. We can rewrite Problem (4.9) as follows:

$$\min_{x \in \mathbb{R}^n} \begin{pmatrix} v \\ x \end{pmatrix}^T \begin{pmatrix} C_* & B_*^T \\ B_* & A_* \end{pmatrix} \begin{pmatrix} v \\ x \end{pmatrix} \quad (4.12)$$

By [Boyd et al. \(2004\)](#), section A.5.5:

$$\begin{pmatrix} C_* & B_*^T \\ B_* & A_* \end{pmatrix} \succeq 0 \iff A_* \succeq 0, \quad B_* v \in \mathcal{R}(A_*), \quad C_* - B_*^T A_*^\dagger B_* \succeq 0 \quad (4.13)$$

Problem (4.8) is a special case of Problem (4.12) when the dimension p of the matrices B_* and C_* is equal to 1 and $v = 1$. It can be rewritten as

$$\text{Min}_{x \in \mathbb{R}^n} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \begin{pmatrix} c(\gamma) & b(\gamma)^T \\ b(\gamma) & A(\gamma) \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}. \quad (4.14)$$

By (4.11), we can rewrite Problem (4.8) as

$$\text{Max}_{\gamma \geq 0} \quad U_0 := -b(\gamma)^T A(\gamma)^\dagger b(\gamma) + c(\gamma). \quad (4.15)$$

In fact, U_0 is the Schur Complement of the quadratic matrix in (4.14).

Consider a new variable $\tau \in \mathbb{R}$ such that $U_0 \geq \tau$.

Let $U_1 := -b(\gamma)^T A(\gamma)^\dagger b(\gamma) + c(\gamma) - \tau$.

Problem (4.15) can be rewritten as:

$$\text{Max}_{\tau, \gamma \geq 0} \quad \tau \quad \text{s.t.} \quad U_1 = U_0 - \tau \geq 0 \quad (4.16)$$

Note that U_1 is the Schur complement of $\begin{pmatrix} c - \tau & b(\gamma) \\ b(\gamma)^T & A(\gamma) \end{pmatrix}$.

Now consider the three conditions on the right-hand side of (4.13):

1. By assumption, $A(\gamma) \succeq 0$.
2. The condition $B_*v \in \mathcal{R}(A_*)$ in (4.13) becomes $\exists x \in \mathbb{R}^n$ such that $A(\gamma)x = b(\gamma)$, which is automatically satisfied by the definition of $A(\gamma)^\dagger$.
3. Note that $U_1 \geq 0$ since $U_0 \geq \tau$.

Hence, the left-hand side of (4.13) holds, i.e. $\begin{pmatrix} c - \tau & b(\gamma)^T \\ b(\gamma) & A(\gamma) \end{pmatrix} \succeq 0$ and it is possible to rewrite (4.16) as (*Dual*₁).

□

An analogous theorem as above can be found in Yamada and Takeda (2018).

Remark 13. From the proof of Theorem 34, it is clear that if $A(\gamma) := A_0 + \sum_{i=1}^n \gamma_i A_i \not\preceq 0$ for every γ , then $\text{Min}_{x \in \mathbb{R}^n} x^T A(\gamma)x + 2b(\gamma)^T x + c(\gamma) = -\infty$. Hence, through all the paper, we make the following assumption:

Assumption 35. There exists $\gamma \geq 0$ such that:

$$A_0 + \sum_{i=1}^n \gamma_i A_i \succeq 0 \tag{4.17}$$

Problem (*Dual*₁) is the most common form in which it is possible to find the Lagrangian Dual of (QCQP) in literature. In the following, we introduce an alternative form for the Lagrangian Dual of (QCQP), the same of Burer and Ye (2020), which is much suitable to study the duality and the exactness of (SDP).

Take $\kappa = -\gamma \leq 0$. Define $S(\kappa) := A_0 - \sum_{i=1}^m \kappa_i A_i$, $s(\kappa) := b_0 - \sum_{i=1}^m \kappa_i b_i$.

$$\begin{aligned}
& \text{Max}_{\lambda \in \mathbb{R}, \kappa \in \mathbb{R}^m} \quad - \sum_{i=1}^m \kappa_i c_i - \lambda \\
& \text{s.t.} \quad \kappa \leq 0 \\
& Z(\lambda, \kappa) := \begin{pmatrix} \lambda & s(\kappa)^T \\ s(\kappa) & S(\kappa) \end{pmatrix} \succeq 0
\end{aligned} \tag{Dual_2}$$

Theorem 36. *The Dual Problem (Dual₂) is equivalent to the Lagrangian Dual Problem (Dual₁), in the sense that the solution sets and the optimal values are the same.*

Proof. Consider the Dual Problem (4.8), i.e. $\max_{\gamma \geq 0} \Phi(\gamma)$. Let $c_0 = 0$ for simplicity. We have:

$$\Phi(\gamma) = x^T (A_0 + \sum_{i=1}^m \gamma_i A_i) x + 2(b_0 + \sum_{i=1}^m \gamma_i b_i)^T x + (\sum_{i=1}^m \gamma_i c_i) \tag{4.18}$$

Set $\kappa = -\gamma$:

$$\begin{aligned}
\Phi(\kappa) &= x^T (A_0 - \sum_{i=1}^m \kappa_i A_i) x + 2(b_0 - \sum_{i=1}^m \kappa_i b_i)^T x - (\sum_{i=1}^m \kappa_i c_i) = \\
&= x^T S(\kappa) x + 2s(\kappa)^T x - \sum_{i=1}^m \kappa_i c_i
\end{aligned} \tag{4.19}$$

At this point, we repeat the same steps shown in Section 3. Since $S(\kappa) \succeq 0$ by assumption, $\min_{x \in \mathbb{R}^n} \Phi(\kappa) = -\sum_{i=1}^m \kappa_i c_i - s(\kappa)^T S(\kappa)^\dagger s(\kappa)$. Hence we introduce a variable $\tau \in \mathbb{R}$ such that $-\sum_{i=1}^m \kappa_i c_i - s(\kappa)^T S(\kappa)^\dagger s(\kappa) \geq \tau$. For $\lambda = -\tau - \sum_{i=1}^m \kappa_i c_i$, we

have:

$$\lambda - s(\kappa)^T S(\kappa)^\dagger s(\kappa) \geq 0 \quad (4.20)$$

which is the Schur complement of $S(\kappa)$ in $Z(\lambda, \kappa)$. Also note that $\tau = -\sum_{i=1}^m \kappa_i c_i - \lambda$ □

4.4.2 SDP duality

In this subsection, we show the connection between the (SDP) relaxation of a (QCQP) and the minimization of a function over a convex cone studied in [Shapiro and Scheinberg \(2000\)](#). This is the preeminent way suggested in literature to prove the zero duality gap and the strong duality between (SDP) and its Lagrangian Dual. However, it may be difficult to find a reference where all the essential steps to unveil this connection are explicitly stated.

Consider the following problem:

$$\begin{aligned} \text{Min}_{\lambda \in \mathbb{R}, \kappa \in \mathbb{R}^m} \quad & \sum_{i=1}^m \kappa_i c_i + \lambda \\ \text{s.t.} \quad & \kappa \leq 0 \\ & Z(\lambda, \kappa) := \begin{pmatrix} \lambda & s(\kappa)^T \\ s(\kappa) & S(\kappa) \end{pmatrix} \succeq 0 \end{aligned} \quad (4.21)$$

Remark 14. Let $\text{Opt}(P)$ be the optimal value of a generic problem P . Since $\max_{x \in \mathcal{F}} f(x) = -\min_{x \in \mathcal{F}} -f(x)$, then $\text{Opt}(\text{Dual}_2) = -\text{Opt}(4.21)$.

Remark 15. Problem (4.21) is a particular case of the following problem studied in

Shapiro and Scheinberg (2000):

$$\text{Min}_{\alpha \in C_1} f(\alpha) \quad \text{s.t.} \quad G(\alpha) \in C_2 \quad (4.22)$$

where C_1 is a closed convex cone in $\mathbb{R}^{m+1}$, $f : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$; C_2 is a closed convex cone in the finite dimensional vector space $\mathcal{Y}$ equipped with the scalar product “ $\cdot$ ” and $G : \mathbb{R}^{m+1} \rightarrow \mathcal{Y}$.

Proof. Since λ is the North West minor of order one of $Z(\lambda, \kappa)$, if $\lambda < 0$, $Z(\lambda, \kappa) \not\preceq 0$. Set $-\lambda = \xi \in \mathbb{R}_-$. In (4.22), take $\alpha = (\xi, \kappa_1, \dots, \kappa_n)$, $C_1 = \mathbb{R}_-^{m+1}$, $C_2 = S_-^{n+1}$. Let $f(\alpha) = \sum_{i=1}^m \kappa_i c_i - \xi$. We can rewrite $G(\alpha) \in S_-^{n+1}$ as $-Z(-\xi, \kappa) \preceq 0$. $\square$

Define the Lagrangian of (4.22) as:

$$L(\alpha, R) := f(\alpha) + R \cdot G(\alpha), \quad (\alpha, R) \in \mathbb{R}^{m+1} \times \mathcal{Y} \quad (4.23)$$

By Shapiro and Scheinberg (2000), Problem (15) is equivalent to the following:

$$\text{Min}_{\alpha \in C_1} \left\{ \text{Max}_{R \in C_2^-} L(\alpha, R) \right\} \quad (4.24)$$

The Lagrangian Dual of (4.22) is:

$$\text{Max}_{R \in C_2^-} \left\{ \text{Min}_{\alpha \in C_1} L(\alpha, R) \right\} \quad (4.25)$$

Theorem 37. *In the case of Problem (4.21), the Lagrangian Dual (4.25) is equivalent*

to the following:

$$\begin{aligned}
& \text{Max}_{X \in S^n, x \in \mathbb{R}^n} -\text{Tr}(XA_0) - 2b_0^T x \\
& \text{s.t. } \text{Tr}(XA_i) + 2b_i^T x + c_i \leq 0, \quad i = 1, \dots, m \\
& R := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0
\end{aligned} \tag{4.26}$$

Proof. We have:

$$\begin{aligned}
\text{Min}_{\alpha \in C_1} L(\alpha, R) &= \text{Min}_{\lambda \in \mathbb{R}, \kappa \in \mathbb{R}^m} \sum_{i=1}^m \kappa_i c_i + \lambda - \text{Tr}(R, Z(\lambda, \kappa)) \\
&\text{s.t. } \kappa \leq 0.
\end{aligned} \tag{4.27}$$

For $\mu_i \geq 0$, $i \in \{1, \dots, m\}$ not all zero, the Lagrangian of (4.27) is:

$$L(\lambda, \kappa, \mu) = \sum_{i=1}^m \kappa_i c_i + \lambda + \mu^T \kappa - \text{Tr}(R, Z(\lambda, \kappa)) \tag{4.28}$$

Take $R = \begin{pmatrix} R_{11} & x^T \\ x & X \end{pmatrix}$ with $X \in S^n$. We recall that $Z(\lambda, \kappa) := \begin{pmatrix} \lambda & s(\kappa)^T \\ s(\kappa) & S(\kappa) \end{pmatrix}$ with $S(\kappa) = A_0 - \sum_{i=1}^m \kappa_i A_i$ and $s(\kappa) = b_0 - \sum_{i=1}^m \kappa_i b_i$. Taking $\gamma = -\kappa$, (4.28) becomes:

$$\begin{aligned}
L(\lambda, \gamma, \mu) &= \\
& - \sum_{i=1}^m \gamma_i c_i + \lambda - \sum_{i=1}^m \gamma_i \mu_i - (\text{Tr}(A_0 X) + 2b_0^T x) - \sum_{i=1}^m \gamma_i (\text{Tr}(A_i X) + 2b_i^T x) - R_{11} \lambda
\end{aligned} \tag{4.29}$$

Consider the gradient of (4.29):

$$\nabla_\lambda = 1 - R_{11} \quad (4.30)$$

$$\nabla_{\kappa_i} = -\text{Tr}(X, A_i) - 2b_i^T x - c_i - \mu_i \quad \forall i \in \{1, \dots, m\} \quad (4.31)$$

Setting (4.31) equal to zero, we get:

$$\text{Tr}(X, A_i) + 2b_i^T x + c_i = -\mu_i \leq 0 \quad \forall i \in \{1, \dots, m\} \quad (4.32)$$

Hence, with $R_{11} = 1$, we can rewrite the Lagrangian Dual (4.25) as (4.26).

□

We make the following assumption in order to present the results in [Shapiro and Scheinberg \(2000\)](#) about the zero duality gap and the strong duality between (4.24) and (4.25):

Assumption 38.

$$0 \in \text{int}\{G(C_1) - C_2\} \quad (4.33)$$

where $G(C_1) = \{y \in \mathcal{Y} \mid \exists x \in C_1 : \text{s.t. } G(x) = y\}$.

The following two theorems establish the conditions under which the zero duality gap and the strong duality hold for Problems (4.24) and (4.25):

Theorem 39. *(Theorem 2.5 from [Shapiro and Scheinberg \(2000\)](#)) Suppose that the primal problem (4.24) is convex and Assumption 38 holds. Then there is zero duality gap between the primal (4.24) and the dual problem (4.25). Moreover, if their*

common optimal value is finite, then the set of solution of (4.25) is nonempty and bounded.

Theorem 40. (Theorem 2.6 from [Shapiro and Scheinberg \(2000\)](#)) Suppose that the primal problem (4.24) is convex and the set of solution of (4.25) is nonempty and bounded. Then Assumption 38 holds and there is zero duality gap between the primal and the dual problem (4.25).

Now focus on Assumption 38:

Definition 41. Given a closed convex cone $C \subset \mathcal{Y}$, we say that $G : \mathbb{R}^{m+1} \rightarrow \mathcal{Y}$ is convex with respect of C if $\forall \alpha_1, \alpha_2 \in \mathbb{R}^{m+1}$ and $\forall t \in [0, 1]$ the following holds:

$$tG(\alpha_1) + (1 - t)G(\alpha_2) \succeq_C G(t\alpha_1 + (1 - t)\alpha_2) \quad (4.34)$$

where “ $\succeq_C$ ” is the partial order associated with C , i.e. $a \succeq_C b$ if $a - b \in C$ ([Shapiro and Scheinberg \(2000\)](#), (2.22)).

In the case of semidefinite programming, $\mathcal{Y} = S^{n+1}$ and $C = S_+^{n+1}$. Considering the Loewner partial ordering as in Definition 32, (4.34) becomes:

$$z^T(tG(\alpha_1) + (1 - t)G(\alpha_2))z \geq z^T G(t\alpha_1 + (1 - t)\alpha_2)z, \forall z \in \mathbb{R}^{n+1} \quad (4.35)$$

([Shapiro and Scheinberg \(2000\)](#), (2.23))

Theorem 42. (Proposition 2.4 from [Shapiro and Scheinberg \(2000\)](#)) Suppose that the mapping $G(\alpha)$ is convex with respect to the cone $-C_2$ and that C_2 has a nonempty

interior. Then Assumption 38 is equivalent to the Slater condition, i.e. there exists $\bar{\alpha}$ such that $G(\bar{\alpha})$ belongs to the interior of $-C_2$.

In the case of Problem (4.21), since the interior of S_+^n is S_{++}^n , the Slater condition is equivalent to:

$$\exists (\bar{\lambda}, \bar{\kappa}) \in \mathbb{R} \times \mathbb{R}^m \text{ s.t. } Z(\bar{\lambda}, \bar{\kappa}) \succ 0 \quad (4.36)$$

Hence, for Problem (4.21) Theorem 39 and Theorem 40 take the following forms:

Theorem 43. *Suppose that the Slater Condition (4.36) holds. Then there is zero duality gap between the primal (4.21) and the dual problem (4.26). Moreover, if their common optimal value is finite, then the set of solution of (4.26) is nonempty and bounded.*

Theorem 44. *Suppose that the set of solution of (4.26) is nonempty and bounded. Then the Slater Condition (4.36) holds and there is zero duality gap between the primal (4.21) and the dual problem (4.26).*

The results of this subsection are summarized by the following remark.

Remark 16.

1. The zero duality gap between (4.21) and (4.26) implies the zero duality gap between (SDP) and ($Dual_2$).
2. If the strong duality holds for (4.21) and (4.26), it also holds for (SDP) and ($Dual_2$).

Proof.

Proof of 1 We have $-Opt(4.21) = Opt(Dual_2)$ and $-Opt(4.26) = Opt(SDP)$.

Proof of 2 It is sufficient to notice that the feasible sets of (4.21) and (4.26) coincide with the feasible sets of respectively $(Dual_2)$ and (SDP) .

□

4.5 Hidden convexity and KKT conditions for QCQPs

Hidden convexity is a term introduced by Ben-Tal and Teboulle (1996). We say (QCQP) is *hidden convex* if it is equivalent to a convex problem (CP), i.e. $\exists (CP)$ such that $opt(QCQP) = opt(CP)$.

By the result presented in Section 4.4, it is clear that a problem is hidden convex if strong duality holds with respect to the Lagrangian dual problem, or, equivalently, if the (SDP) relaxation is exact.

Consider a QCQP problem which satisfies the following assumption.

Assumption 45. Assume that there exists a global minimum x^* of problem (QCQP) and there exists $\gamma \in \mathbb{R}_+^m \setminus \mathbf{0}_m$ such that $A_J + \sum \gamma_k A_k \succeq 0$.

As in Theorem 34, let for any $\gamma \in \mathbb{R}_+^m$, $A(\gamma) := A_0 + \sum_{i=1}^m \gamma_i A_i$, $b(\gamma) := b_0 + \sum_{i=1}^m \gamma_i b_i$, $c(\gamma) := c_0 + \sum_{i=1}^m \gamma_i c_i$. We define the Lagrangian function $L(x, \gamma)$ of (QCQP) as follow

$$L(x, \gamma) := x^T A(\gamma) x + 2b(\gamma)^T x + c(\gamma). \quad (4.37)$$

The following theorem unveils the connection between the [SDP](#) relaxation and the Lagrangian Duality for [\(QCQP\)](#). An analogous result is Proposition 2 from [Bao et al. \(2011\)](#).

Theorem 46. *Consider a Problem [\(QCQP\)](#) which satisfies Assumption [45](#) and there exists a point $x_0 \in \mathbb{R}^n$ for all $f_k(x_0) < 0$, $k = 1, \dots, m$. Consider the relaxations [SDP](#) and [Dual₂](#) of Problem [\(QCQP\)](#) and assume that such that Slater condition [\(4.36\)](#) holds for [Dual₂](#). Then, the following chain of equations and inequalities holds.*

$$\begin{aligned} \text{Opt}(\text{QCQP}) &= \text{Min}_{x \in \mathbb{R}^n} \text{Max}_{\gamma \geq 0} L(x, \gamma) \geq \text{Max}_{\gamma \geq 0} \text{Min}_{x \in \mathbb{R}^n} L(x, \gamma) \\ &= \text{Opt}(\text{Dual}_1) = \text{Opt}(\text{Dual}_2) = \text{Opt}(\text{SDP}). \end{aligned} \tag{4.38}$$

Proof. The first equality and the inequality trivially hold, the second equality holds due to Theorem [34](#), the third equality holds by Theorem [36](#) and the forth equality holds by Remark [16](#). $\square$

The next theorem states the equivalence between the exactness of the [SDP](#) relaxation and the global minima characterization via KKT conditions for problem [QCQP](#).

Theorem 47. *Under the assumptions of Theorem [46](#), the following two statements are equivalent.*

1. *The KKT conditions are necessary and sufficient for [\(QCQP\)](#).*
2. *The [\(SDP\)](#) is exact (or tight).*

Proof. Statement (1) is equivalent to $\exists (x^*, \gamma^*) \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$\text{Min}_{x \in \mathbb{R}^n} \text{Max}_{\gamma \geq 0} L(x, \gamma) = L(x^*, \gamma^*) = \text{Max}_{\gamma \geq 0} \text{Min}_{x \in \mathbb{R}^n} L(x, \gamma).$$

Statement (2) is equivalent to $\text{Opt}(\text{QCQP}) = \text{Opt}(\text{SDP})$. The proof follows directly from Theorem 46 . $\square$

4.6 The S-lemma

The first result on hidden convexity was first proved (ante litteram) for (QCQP) problems with only one constraint, i.e. $m = 1$, thanks to the celebrated S-Lemma, see e.g. Yakubovich (1992).

The results presented in this section are valid in Hilbert spaces.

Problems of the form (QCQP) in Hilbert spaces arise in calculus of variations Hestenes (1951), in optimal control Yakubovich (1992) and in the context of reproducing kernel Hilbert space Signoretto et al. (2008).

The key step in the proof of the S-lemma is the convexity of the *Generalized Image Set*, c.f. Jeyakumar et al. (2009).

$$\text{GIS} := \{(f_0(x), f_1(x), \dots, f_m(x)) \mid x \in H\} + \text{int} \mathbb{R}_+^{m+1}, \quad (\text{GIS})$$

$f_0(x) := J(x) - J^*$ and $'+'$ is the Minkowski sum of the image set

$$\{(f_0(x), f_1(x), \dots, f_m(x)) \mid x \in H\} \quad (4.39)$$

and the interior of the cone $\mathbb{R}_+^{m+1}$. By [Holmes \(2012\)](#), exercise 2.1, the set Ω_0 is open. Recall, that the image set of optimization problems defined in (4.39) has been extensively investigated in the monograph [Giannessi \(2006\)](#).

The S-Lemma is an important result related to the general S-procedure described in [Fradkov and Yakubovich \(1979\)](#), treated from an historical point view in [Gusev and Likhtarnikov \(2006\)](#). Paper [Pólik and Terlaky \(2007\)](#) provides a comprehensive survey on the S-Lemma in $\mathbb{R}^n$, while [Jeyakumar et al. \(2007\)](#) explores the relations between the S-Lemma and the Lagrangian multipliers of QCQP. In [Barro et al. \(2021\)](#), a S-Lemma based approach, similar to the one proposed by [Jeyakumar et al. \(2009\)](#), is applied to problem with data uncertainty in $\mathbb{R}^n$. The paper [Ruiz Galán \(2017\)](#) extends the results of [Jeyakumar et al. \(2009\)](#) to Z-matrices QCQP in $\mathbb{R}^n$ with infinite number of inequalities. S-Lemma and SDP relaxation for the trust region problem, an intensively discussed class of QCQP, is considered [Jeyakumar and Li \(2014\)](#). More recent generalization of the S Lemma in $\mathbb{R}^n$ appear in [Xia et al. \(2016\)](#), [Song and Xia \(2022\)](#), where the sets (4.39) and GIS play a significant role.

For (QCQP) with one or two constraints, i.e. $m = 1$ and $m = 2$, the convexity of (4.39) can be proved under some regularity conditions, starting from the results of Dine and Polyak (see [Jeyakumar et al. \(2009\)](#); [Pólik and Terlaky \(2007\)](#)). For Z-matrices QCQP, the convexity of GIS is proved in [Jeyakumar et al. \(2009\)](#).

In [Contino et al. \(2022\)](#), the results of Dine and Polyak are generalized to Hilbert spaces. As a consequence, a generalized form of the S-Lemma for three homogeneous quadratic functionals, i.e. $f_k(x) := \langle x, A_k x \rangle$, $k = 0, 1, 2$, holds in Hilbert spaces.

Throughout the whole section, we are making the following existence assumption.

Assumption 48. There exists a global minimum x^* of problem [QCQP](#) and there exists $\gamma \in \mathbb{R}_+^m \setminus 0$ such that $A_J + \sum \gamma_k A_k$ is nonnegative.

The assumption that there exists a global minimum x^* of problem [QCQP](#), is made to avoid discussing the existence of solution for [QCQP](#) problems in Hilbert spaces, the topic which is covered by other papers, (cf. [Dong and Tam, 2018](#)). For example, if the objective function of [S-QCQP](#) is $J(x) := \|x - z\|^2$, $z \in H$, the existence of solution is guaranteed.

For convex QP problems (quadratic convex objective functions under linear constraints only) in Hilbert spaces, conditions for the existence of solutions and global optimality conditions can be found respectively in Theorem 3.128 and in Theorem 3.130 of [Bonnans and Shapiro \(2013\)](#).

4.6.1 Generalized S-Lemma, convex separation theorem

Consider a collection of quadratic functionals $f_k : H \rightarrow \mathbb{R}$ ($k = 1, \dots, m$). In the spirit of [Pólik and Terlaky \(2007\)](#), a theorem of the alternative is called generalized version of the S-Lemma if it establishes the conditions on the functionals f_k under which only one between the following statements holds:

1. $\exists x \in H$ such that $f_k(x) < 0 \quad \forall k \in \{1, \dots, m\}$
2. $(\exists \gamma \in \mathbb{R}_+^m \setminus 0) \sum_{k=1}^m \gamma_k f_k(x) \geq 0 \quad \forall x \in H$

Let C_1 and C_2 be non-empty subsets in $\mathbb{R}^n$.

Definition 49. ([Rockafellar, 1997](#), section 11)

- A hyperplane P is said to *separate* C_1 and C_2 if C_1 is contained in one of the closed half spaces associated to P and C_2 is contained in the opposite closed half space.
- P is said to *separate properly* C_1 and C_2 if they are not both contained in P itself.

For n -dimensional convex sets in $\mathbb{R}^n$, $\text{aff} C = \mathbb{R}^n$ and so, by (1.3), we have $\text{ri} C = \text{int} C$ (Rockafellar, 1997, section 6). Hence, we can rewrite (Rockafellar, 1997, Theorem 11.3) as follows.

Theorem 50. *Let C_1 and C_2 be n -dimensional non-empty convex sets in $\mathbb{R}^n$. In order that there exists a hyperplane that separates C_1 and C_2 properly, it is necessary and sufficient that $\text{int} C_1$ and $\text{int} C_2$ have no point in common.*

4.6.2 Direct sum decompositions in Hilbert spaces

We apply the following concepts and results in section 4. This subsection is based on the monograph Deutsch and Deutsch (2001).

$H_1 \oplus H_2$ denotes the direct sum of H_1 and H_2 . If H_1 is a closed subspace of H , one can always find a subspace H_2 such that $H = H_1 \oplus H_2$, H_2 is called a complement to H_1 and is orthogonal to H_1 , $H_2 = H_1^\perp$, i.e. $\langle x_1, x_2 \rangle = 0$, for any $x_1 \in H_1$, $x_2 \in H_2$.

Definition 51. (Deutsch and Deutsch, 2001, Definition 7.9) Let H be a inner product space. A closed subspace H_1 of H is said to have codimension $m < \infty$, written $\text{codim } H_1 = m$, if there exists a subspace H_2 with $\dim H_2 = m$ such that $H = H_1 \oplus H_2$.

The following result is a consequence of Theorem 7.11 and Theorem 7.12 from [Deutsch and Deutsch \(2001\)](#).

Theorem 52. *Let H be a Hilbert space. Let V be a closed subspace of H and m be a positive integer. Then, $\text{codim } V = m$ if and only if*

$$V = \bigcap_{i=1}^m \{x \in H \mid \langle b_i, x \rangle = 0\},$$

for some linearly independent set $\{b_1, b_2, \dots, b_m\}$ in H .

By ([Rudin, 1991](#), Theorem 12.10), for any continuous linear operator $A : H \rightarrow H$ we have

$$H = \text{range } A \oplus \ker A^*, \quad H = \ker A \oplus \text{range } A^* \quad (4.40)$$

4.6.3 Global Minima Characterization for general QCQP

In this section we characterize global minima of QCQP problem by (KKT) conditions derived with the help of a generalized form of the S-Lemma as defined in Section 2.

Our approach is inspired by the one proposed in [Jeyakumar et al. \(2009\)](#) to characterize, in $\mathbb{R}^n$, the global minima of Z -matrices QCQP, i.e. QCQP with the matrices

$$H_k := \begin{pmatrix} A_k & b_k \\ b_k^T & c_k \end{pmatrix} \quad k = 1, \dots, m \text{ and } H_J := \begin{pmatrix} A_J & b_J \\ b_J^T & c_J \end{pmatrix} \quad (4.41)$$

having all the off diagonal elements nonpositive.

Consider a collection of quadratic functionals $f_k : H \rightarrow \mathbb{R}$, $k = 0, 1, \dots, m$ defined

on a Hilbert space H ,

$$f_k(x) := \langle x, A_k x \rangle + 2\langle b_k, x \rangle + c_k, \quad (4.42)$$

where $A_k : H \rightarrow H$ are self-adjoint linear continuous operators acting on the space H and $b_k \in H$, $k = 0, 1, \dots, m$.

Theorem 53. *Let H be a Hilbert space and let $f_k : H \rightarrow \mathbb{R}$, $k = 0, 1, \dots, m$ be a finite collection of functionals of the form (4.42). Consider the generalized image set GIS , defined as*

$$GIS := \{(f_0(x), f_1(x), \dots, f_m(x)) \mid x \in H\} + \text{int}\mathbb{R}_+^{m+1}.$$

Assume that the set GIS is convex. Then, exactly one of the following statements is valid:

$$(i) \quad \exists \bar{x} \in H \text{ such that } f_k(\bar{x}) < 0 \quad \forall k \in \{0, \dots, m\},$$

$$(ii) \quad (\exists \gamma \in \mathbb{R}_+^{m+1} \setminus \mathbf{0}_{m+1}) \sum_{k=0}^m \gamma_k f_k(x) \geq 0 \quad \forall x \in H.$$

Proof. The implication $[\text{not}(ii) \Rightarrow (i)]$ is immediate (by contradiction). To show the implication $[\text{not}(i) \Rightarrow (ii)]$, assume that (i) does not hold, i.e., the system

$$f_k(x) < 0 \quad \forall k \in \{0, \dots, m\} \quad (4.43)$$

has no solution. By the definition of GIS , the inconsistency of the system (4.43)

implies that

$$\text{GIS} \cap (-\text{int}\mathbb{R}_+^{m+1}) = \emptyset. \quad (4.44)$$

To see this, suppose, by contrary, that there exists $y \in \text{GIS} \cap (-\text{int}\mathbb{R}_+^{m+1})$. The definition of GIS (with f_k , $k = 0, \dots, m$, defined by (4.42)) implies that there exist $x_0 \in H$, $C_0, C_1 \in \text{int}\mathbb{R}_+^{m+1}$ such that

$$y_0 = (f_0(x_0), \dots, f_m(x_0)) + C_0 = -C_1 \in (-\text{int}\mathbb{R}_+^{m+1}),$$

i.e., $(f_0(x_0), \dots, f_m(x_0)) = -C_0 - C_1 \in (-\text{int}\mathbb{R}_+^{m+1})$ contradictory to (4.43). This proves (4.44).

Since int GIS and $-\text{int}\mathbb{R}_+^{m+1}$ are $(m+1)$ -dimensional sets in $\mathbb{R}^{m+1}$, non-empty and convex, by (4.44),

$$\text{int GIS} \cap (-\text{int}\mathbb{R}_+^{m+1}) = \emptyset,$$

we can apply Theorem 50. So, there exists a hyperplane which separates GIS and $-\text{int}\mathbb{R}_+^{m+1}$ properly, i.e. there exists $\gamma = (\gamma_0, \dots, \gamma_m) \in \mathbb{R}^{m+1} \setminus 0$ such that

$$\sum_{k=0}^m \gamma_k y_k \geq 0 \quad \forall y = (y_0, \dots, y_m) \in \text{GIS} \quad (4.45)$$

and $\sum_{k=0}^m \gamma_k z_k \leq 0 \quad \forall z = (z_0, \dots, z_m) \in (-\text{int}\mathbb{R}_+^{m+1})$. This latter inequality implies that it must be $\gamma \in \mathbb{R}_+^{m+1} \setminus 0$.

Consequently, by the definition of GIS,

$$y = (f_0(x), \dots, f_m(x)) + C \in \text{GIS}, \quad C = (C_0, \dots, C_m) \in \text{int}\mathbb{R}_+^{m+1}, \quad x \in H$$

and by the formula (4.45), we get

$$\sum_{k=0}^m \gamma_k (f_k(x) + C_k) \geq 0 \quad \forall x \in H, \quad C_k \in \text{int}\mathbb{R}_+ \quad k = 0, \dots, m. \quad (4.46)$$

We show that it must be

$$\sum_{k=0}^m \gamma_k f_k(x) \geq 0 \quad \forall x \in H. \quad (4.47)$$

Otherwise, $\sum_{k=0}^m \gamma_k f_k(\bar{x}) < 0$ for some $\bar{x} \in H$, and it would be possible to choose $C \in \text{int}\mathbb{R}_+^{m+1}$ with components $C_k > 0$ small enough so as

$$\sum_{k=0}^m \gamma_k (f_k(\bar{x}) + C_k) < 0$$

which would contradict (4.45) since $(f_0(\bar{x}) + \ell_0, \dots, f_m(\bar{x}) + \ell_m) \in \Omega_0$ for any $\ell = (\ell_0, \dots, \ell_m) \in \text{int}\mathbb{R}_+^{m+1}$. Thus, (4.47) holds and it proves (ii). $\square$

In the following, we exploit Theorem 53 to get necessary and sufficient optimality conditions for general QCQP.

Theorem 54. *Let H be a Hilbert space. Let Assumption 45 hold and x^* be a global minimizer of QCQP. Define*

$$f_0(x) := J(x) - J(x^*) = \langle x, A_J x \rangle + 2\langle b_J, x \rangle - \langle x^*, A_J x^* \rangle - 2\langle b_J, x^* \rangle.$$

Consider the collection of quadratic functionals formed by $f_0(x)$ and the constraints of (QCQP) $f_i(x)$, $i = 1, \dots, m$ given by (4.42). Let GIS be convex. The following

Fritz-John conditions are necessary for global optimality of a feasible x^* to [QCQP](#), i.e. there exists a vector $(\gamma_0, \dots, \gamma_m) \in \mathbb{R}_+^{m+1} \setminus 0$ such that

$$\begin{aligned} (i) \quad & \nabla(\gamma_0 J + \sum_{k=1}^m \gamma_k f_k)(x^*) = 0, \\ (ii) \quad & \gamma_k f_k(x^*) = 0 \quad k = 1, \dots, m, \\ (iii) \quad & \gamma_0 A_J + \sum_{k=1}^m \gamma_k A_k \text{ is nonnegative.} \end{aligned} \tag{4.48}$$

Moreover, if there exists a point $x_0 \in H$ such that

$$f_k(x_0) < 0 \quad \forall k \in \{1, \dots, m\}, \tag{4.49}$$

then there exists a vector $(\gamma_1, \dots, \gamma_m) \in \mathbb{R}_+^m \setminus 0$ such that

$$\begin{aligned} (i) \quad & \nabla(J + \sum_{k=1}^m \gamma_k f_k)(x^*) = 0, \\ (ii) \quad & \gamma_k f_k(x^*) = 0 \quad k = 1, \dots, m, \\ (iii) \quad & A_J + \sum_{k=1}^m \gamma_k A_k \text{ is nonnegative.} \end{aligned} \tag{KKT}$$

are necessary for optimality. Given a feasible x^* for problem [QCQP](#), if (4.49) holds, the conditions (KKT) are also sufficient for global optimality of x^* .

Proof. Let $f_0(x) := J(x) - J(x^*)$. Since x^* is a global minimizer of [QCQP](#), $f_0(x) \geq 0 \quad \forall x$ feasible for [QCQP](#). Hence, the system $f_k(x) < 0 \quad k = 0, \dots, m$ has no solution. By

Theorem 53, there exists $(\gamma_0, \dots, \gamma_m) \in \mathbb{R}_+^{m+1} \setminus 0$ such that

$$\gamma_0 f_0(x) + \sum_{k=1}^m \gamma_k f_k(x) \geq 0 \quad \text{for all } x \in H. \quad (4.50)$$

In particular, for $x = x^*$, we have $\sum_{k=1}^m \gamma_k f_k(x^*) \geq 0$. Since $\gamma_k f_k(x^*) \leq 0 \quad \forall k \in \{1, \dots, m\}$, it must be $\gamma_k f_k(x^*) = 0 \quad \forall k \in \{1, \dots, m\}$ which proves (ii) of (4.48). Moreover, by (4.50) and (ii) of (4.48), for all $x \in H$

$$\tilde{L}(x) := \gamma_0 J(x) + \sum_{k=1}^m \gamma_k f_k(x) \geq \gamma_0 J(x^*) = \tilde{L}(x^*). \quad (4.51)$$

Hence, $\tilde{L}(x)$ attains its minimum over H at x^* . Notice that we can rewrite $\tilde{L}(x)$ as

$$\tilde{L}(x) := \gamma_0 J(x) + \sum_{k=1}^m \gamma_k f_k(x) = \langle x, \tilde{A}(\gamma)x \rangle + 2\langle \tilde{b}(\gamma), x \rangle + \tilde{c}(\gamma) \quad (4.52)$$

with $\tilde{A}(\gamma) = \gamma_0 A_J + \sum_{k=1}^m \gamma_k A_k$, $\tilde{b}(\gamma) = \gamma_0 b_J + \sum_{k=1}^m \gamma_k b_k$ and $\tilde{c}(\gamma) = \gamma_0 c_J + \sum_{k=1}^m \gamma_k c_k$.

The operator $\tilde{A}(\gamma)$ must be nonnegative, otherwise we would have

$$\inf_{x \in H} \tilde{L}(x) = -\infty,$$

contradictory to (4.51). Hence, condition (iii) of (4.48) holds and $\tilde{L}(x)$ is convex with respect to x . We can apply Fermat's optimality condition, (cf. Bauschke and Combettes, 2017b, Theorem 16.3), to the convex and twice continuously differentiable function $\tilde{L}(x)$. The optimality condition $\nabla_x \tilde{L}(x^*) = 0$ is equivalent to the conditions (i) of (4.48). The proof of the necessity part of the theorem is finished.

Suppose now that (4.49) holds, i.e., there exists a point $x_0 \in H$ such that $f_k(x_0) < 0 \forall k = 1, \dots, m$. If it were $\gamma_0 = 0$, then by (4.50), it would be $\sum_{k=1}^m \gamma_k f_k(x) \geq 0$ for all $x \in H$, which would contradict (4.49). Hence, $\gamma_0 > 0$ and the Fritz-John conditions become the (KKT) conditions, i.e. (KKT) holds.

To complete the proof, we show that conditions (KKT) are also sufficient for optimality. Assume that there exists $x^* \in H$ which is feasible to QCQP and $(\gamma_1, \dots, \gamma_m) \in \mathbb{R}_+^m \setminus 0$ such that (KKT) holds. The Lagrangian for QCQP is:

$$L(x, \gamma) := J(x) + \sum_{k=1}^m \gamma_k f_k(x) = \langle x, A(\gamma)x \rangle + 2\langle b(\gamma), x \rangle + c(\gamma) \quad (4.53)$$

with $A(\gamma) = A_J + \sum_{k=1}^m \gamma_k A_k$, $b(\gamma) = b_J + \sum_{k=1}^m \gamma_k b_k$ and $c(\gamma) = c_J + \sum_{k=1}^m \gamma_k c_k$.

Notice that the Lagrangian $L(x, \gamma)$ is convex with respect to x , since $A(\gamma) = A_J + \sum_{k=1}^m \gamma_k A_k$ is nonnegative by (KKT). Hence, x^* such that $\nabla_x L(x^*) = 0$ is the minimum of $L(x, \gamma)$ for γ fixed, thanks to Fermat's optimality condition, (cf. Bauschke and Combettes, 2017b, Theorem 16.3).

By (KKT), $\gamma_k f_k(x^*) = 0$ for $k = 1, \dots, m$. We have

$$J(x) + \sum_{k=1}^m \gamma_k f_k(x) \geq J(x^*) + \sum_{k=1}^m \gamma_k f_k(x^*) = J(x^*) \quad \forall x \in H. \quad (4.54)$$

For any x feasible for QCQP, $f_k(x) \leq 0$, $k = 1, \dots, m$, and hence

$$J(x) \geq J(x) + \sum_{k=1}^m \gamma_k f_k(x). \quad (4.55)$$

Combining (4.54) and (4.55), for any x feasible for QCQP, we have

$$J(x) \geq J(x^*) \quad (4.56)$$

which proves that x^* is a global minimum for QCQP. $\square$

Example 55. Consider a convex (QCQP) with m constraints and let x^* be the global minimum. It is possible to show that GIS is convex. Fix $\lambda \in (0, 1]$. Take $v := (v_0, \dots, v_m) \in \text{GIS}$, $w := (w_0, \dots, w_m) \in \text{GIS}$, i.e. there exists $x_v, x_w \in H$ such that

$$f_k(x_v) < v_k, \quad f_k(x_w) < w_k, \quad \forall k \in \{0, \dots, m\}. \quad (4.57)$$

We show that $\lambda v + (1 - \lambda)w \in \text{GIS}$, i.e. there exists $\tilde{x} \in H$ such that

$$f_k(\tilde{x}) \leq \lambda f_k(x_v) + (1 - \lambda)f_k(x_w) < \lambda v_k + (1 - \lambda)w_k, \quad \forall k \in \{0, \dots, m\}, \quad (4.58)$$

where the strict inequality is a consequence of (4.57). We choose $\tilde{x} = \lambda x_v + (1 - \lambda)x_w$.

By the convexity of functions $f_k(\cdot)$, for all $k \in \{0, \dots, m\}$

$$f_k(\tilde{x}) = f_k(\lambda x_v + (1 - \lambda)x_w) \leq \lambda f_k(x_v) + (1 - \lambda)f_k(x_w).$$

So, (4.58) holds and GIS is convex. By Theorem 54, there exists a vector $(\gamma_0, \dots, \gamma_m) \in \mathbb{R}_+^{m+1} \setminus 0$ such that the conditions (4.48) are necessary for optimality. Moreover, if there exists a point $x_0 \in H$ such that (4.49) holds, then there exists a vector $(\gamma_1, \dots, \gamma_m) \in \mathbb{R}_+^m \setminus 0$ such that (KKT) are necessary and sufficient for global optimality.

Example 56. Consider problem (QCQP) with one constraint, $m = 1$, defined on a real Hilbert space H of dimension $\dim H$ such that $3 \leq \dim H \leq \infty$. Assume that there exists $\gamma_0, \gamma_1 \in \mathbb{R}$ such that $(\gamma_0 A_0 + \gamma_1 A_1)$ is positive. Then, GIS is convex by Contino et al. (2022), Theorem 4.1. If there exists a global minimum x^* of the considered (QCQP) with $m = 1$, we can apply Theorem 54, i.e. there exists a vector $(\gamma_0, \gamma_1) \in \mathbb{R}_+^2 \setminus 0$ such that the conditions (4.48) are necessary for optimality. Moreover, if there exists a point $x_0 \in H$ such that (4.49) holds, then there exists a scalar $\gamma_1 > 0$ such that (KKT) are necessary and sufficient for global optimality.

Remark 17. In Theorem 54, we use (4.49) as a “constraint qualification”, in the sense that (4.49) allow us to take $\gamma_0 > 0$ in (4.48). In analogy with convex nonlinear programming, it is possible to replace (4.49) with other constraint qualification. For example, consider convex QCQP with linear constraints only (Convex QP). By Theorem 3.118 of Bonnans and Shapiro (2013), if x_1 is a local solution of a Convex QP, there exists a vector $(\gamma_1, \dots, \gamma_m) \in \mathbb{R}_+^m \setminus 0$ such that conditions (i) and (ii) of (KKT) hold in x_1 . Since (iii) of (KKT) holds automatically, Theorem 54 can be proved even without assuming (4.49) for Convex QP.

4.7 Global minima characterization for (S-QCQP)

In the present section we use the results of Section 4.6.3 to provide a characterization of global minima for S-QCQP. The main result of this section is Theorem 57.

Let x^* be the global minimum of S-QCQP. As in Theorem 54, we use the notation

$$f_0(x) := J(x) - J(x^*) = a_J \|x\|^2 + \langle b_J, x \rangle - a_J \|x^*\|^2 - 2\langle b_J, x^* \rangle.$$

Also in the case of [S-QCQP](#), [GIS](#) takes the form

$$\text{GIS} := \{(f_0(x), f_1(x), \dots, f_m(x)) | x \in H\} + \text{int}\mathbb{R}_+^{m+1}$$

where

$$f_k(x) := a_k \|x\|^2 + 2\langle b_k, x \rangle + c_k \quad k = 0, \dots, m$$

with $a_k \in \mathbb{R}$.

Theorem 57. *Consider problem [S-QCQP](#) with m constraints in a Hilbert space H . Suppose that one of the following holds:*

- i. H is infinite dimensional,*
- ii. H has n -dimensional, $n \in \mathbb{N}$. In this case, any maximally linearly independent subset of the system $\langle b_k, x \rangle = 0$, $k = 0, \dots, m$ has cardinality $\overline{m} < n$.*

Then, the generalized image set GIS , is convex.

Proof. In order to show that $\text{GIS} \subset \mathbb{R}^{m+1}$ is convex, take any $v := (v_0, \dots, v_m)$, $w := (w_0, \dots, w_m) \in \text{GIS}$ and $\lambda \in (0, 1)$. There exist $x_v, x_w \in H$ such that

$$f_k(x_v) < v_k \text{ and } f_k(x_w) < w_k \quad \forall k \in \{0, \dots, m\}. \quad (4.59)$$

Consider the convex combination $\lambda v + (1 - \lambda)w$. Let $(\lambda v + (1 - \lambda)w)_k$ be the k -th component of $\lambda v + (1 - \lambda)w$. By [\(4.59\)](#), we have

$$\lambda f_k(x_v) + (1 - \lambda)f_k(x_w) < (\lambda v + (1 - \lambda)w)_k \quad \forall k \in \{0, \dots, m\}. \quad (4.60)$$

In order to prove that GIS is convex, we show that the convex combination $\lambda v + (1 - \lambda)w$ belongs to GIS, i.e. there exists $\tilde{x} \in H$ such that $\forall k \in \{0, \dots, m\}$

$$f_k(\tilde{x}) < (\lambda v + (1 - \lambda)w)_k. \quad (4.61)$$

Formulas (4.60) and (4.61) together imply that GIS is convex if there exists $\tilde{x} \in H$ such that $\forall k \in \{0, \dots, m\}$

$$f_k(\tilde{x}) \leq \lambda f_k(x_v) + (1 - \lambda)f_k(x_w). \quad (4.62)$$

Note that if $x_v = x_w$, then (4.62) trivially holds for $\tilde{x} = x_v = x_w$. From now on, we assume that x_v and x_w are distinct vectors, $x_v \neq x_w$.

Let us consider

$$\mathcal{S}(\lambda) := \{x \in H \mid \|x\|^2 = \lambda\|x_v\|^2 + (1 - \lambda)\|x_w\|^2\}, \quad (4.63)$$

i.e. a sphere centered at zero with radius $\sqrt{\lambda\|x_v\|^2 + (1 - \lambda)\|x_w\|^2}$. In particular, when both $x_v = x_w = 0$ the set $\mathcal{S}(\lambda)$ reduces to $\{0\}$, but this is impossible, since we assumed that x_v and x_w are distinct vectors.

The idea of proving the existence of $\tilde{x}$ satisfying (4.61) is based on the following observations:

– any $\tilde{x} \in \mathcal{S}(\lambda)$ satisfies (4.61) if $\forall k \in \{0, \dots, m\}$:

$$\begin{aligned} f_k(\tilde{x}) &\leq \lambda f_k(x_v) + (1 - \lambda)f_k(x_w) < \lambda v_k + (1 - \lambda)w_k. \\ a_k \|\tilde{x}\|^2 + 2\langle b_k, \tilde{x} \rangle + c_k &\leq \\ a_k(\lambda \|x_v\|^2 + (1 - \lambda)\|x_w\|^2) + 2\langle b_k, (\lambda x_v + (1 - \lambda)x_w) \rangle + c_k, \end{aligned} \quad (4.64)$$

– to ensure (4.64), it is enough to show that we can choose $\tilde{x} \in \mathcal{S}(\lambda)$ satisfying (4.63) and such that $\forall k \in \{0, \dots, m\}$

$$\begin{aligned} \langle b_k, \tilde{x} \rangle &= \langle b_k, (\lambda x_v + (1 - \lambda)x_w) \rangle, \text{ i.e.,} \\ \langle b_k, (\tilde{x} - (\lambda x_v + (1 - \lambda)x_w)) \rangle &= 0. \end{aligned} \quad (4.65)$$

Let V be the set of the solutions of (4.65). Assume that

$$V \cap \mathcal{S}(\lambda) \neq \emptyset \quad (4.66)$$

and take $\tilde{x} \in V \cap \mathcal{S}(\lambda)$. Then, we have for all $k \in \{0, \dots, m\}$

$$\begin{aligned} f_k(\tilde{x}) &= a_k \|\tilde{x}\|^2 + 2\langle b_k, \tilde{x} \rangle + c_k = \\ a_k(\lambda \|x_v\|^2 + (1 - \lambda)\|x_w\|^2) + 2\langle b_k, (\lambda x_v + (1 - \lambda)x_w) \rangle + c_k &= \\ \lambda(a_k \|x_v\|^2 + 2\langle b_k, x_v \rangle + c_k) + (1 - \lambda)(a_k \|x_w\|^2 + 2\langle b_k, x_w \rangle + c_k) &= \\ \lambda f_k(x_v) + (1 - \lambda)f_k(x_w), \end{aligned} \quad (4.67)$$

which proves Theorem 57.

Hence, in order to finish the proof, we need to show that (4.66) holds. We change variable by setting

$$y = \tilde{x} - (\lambda x_v + (1 - \lambda)x_w). \quad (4.68)$$

The sets $\mathcal{S}(\lambda)$ and V become

$$\begin{cases} \mathcal{S}(\lambda) := \{y \in H \mid \|y + (\lambda x_v + (1 - \lambda)x_w)\|^2 = \lambda\|x_v\|^2 + (1 - \lambda)\|x_w\|^2\}, \\ V := \text{solution set of } \langle b_k, y \rangle = 0 \quad \forall k \in \{0, \dots, m\}. \end{cases} \quad (4.69)$$

Notice that, in the space of the variable $y \in H$, $\mathcal{S}(\lambda)$ is a sphere centered in $(-\lambda x_v - (1 - \lambda)x_w)$ of radius $\sqrt{\|x_v\|^2 + (1 - \lambda)\|x_w\|^2}$, while V is the space of solutions of the homogeneous system of equations of the form

$$\langle b_k, y \rangle = 0 \quad k = 0, \dots, m. \quad (4.70)$$

We choose a maximally linearly independent subset of cardinality $\overline{m}$ of the vectors b_k , $k \in \{0, \dots, m\}$. Without loss of generality, we will indicate the vectors in the maximally linearly independent subset with b_k , $k \in \{1, \dots, \overline{m}\}$. Then, the solution set V of system (4.70), coincides with the kernel of the linear continuous operator $\mathcal{B} : H \rightarrow \mathbb{R}^{\overline{m}}$, defined as

$$\mathcal{B}y := \begin{bmatrix} \langle b_1, y \rangle \\ \vdots \\ \langle b_{\overline{m}}, y \rangle \end{bmatrix}. \quad (4.71)$$

The operator $\mathcal{B}$ is continuous because, by the Riesz theorem, each functional $\langle b_k, y \rangle$, $k = 1, \dots, \overline{m}$ is continuous on H . Notice that the range of the operator $\mathcal{B}$ is

finite-dimensional, i.e., $\text{range}(\mathcal{B}) = \mathbb{R}^{\overline{m}}$.

By Theorem 52 and (4.40), $\ker \mathcal{B} = V = \bigcap_{k=1}^{\overline{m}} \{x \in H \mid \langle b_k, x \rangle = 0\}$, for the linearly independent set $\{b_1, b_2, \dots, b_{\overline{m}}\}$ in H . By (4.40), we have

$$H = \ker \mathcal{B} \oplus \text{range } \mathcal{B}^* = V \oplus \text{range } \mathcal{B}^*. \quad (4.72)$$

By Theorem 52, V is a subspace of codimension $\overline{m}$. By the definition of codimension, $\dim \text{range } \mathcal{B}^* = \overline{m}$.

We prove that $V \neq \{0\}$, in the case that H is infinite dimensional, under assumption *i.*, and in the case where H is finite dimensional with $n > \overline{m}$, under assumption *ii.* In case that H is infinite dimensional, it must be $V \neq \{0\}$ since (4.72) holds and $\dim \text{range } \mathcal{B}^* = \overline{m} < \infty$. In the case where H is finite dimensional and $n > \overline{m}$, $V \neq \{0\}$ by (4.72) and $\dim \text{range } \mathcal{B}^* = \overline{m} < n$.

Hence, there exists a point $y_v \in V$ of norm $\|y_v\| > 0$.

We prove that there exists $\tilde{y} \in V \cap \mathcal{S}(\lambda)$ by finding $\alpha \in \mathbb{R}$ such that $\tilde{y} := \alpha y_v \in V \cap \mathcal{S}(\lambda)$.

We always have $\tilde{y} := \alpha y_v \in V$, since, $\forall k \in \{1, \dots, \overline{m}\}$, $\langle b_k, \tilde{y} \rangle = \alpha \langle b_k, y_v \rangle = 0$.

Recall that $\mathcal{S}(\lambda)$ is the sphere defined in (4.69). Hence, in order to prove that $\tilde{y} \in \mathcal{S}(\lambda)$, there should exist $\alpha \in \mathbb{R}$ such that

$$\begin{aligned} \tilde{y} &= \alpha y_v \in \mathcal{S}(\lambda), \text{ i.e.} \\ \|\alpha y_v + \lambda x_v + (1 - \lambda)x_w\|^2 &= \lambda \|x_v\|^2 + (1 - \lambda) \|x_w\|^2. \end{aligned} \quad (4.73)$$

Consequently, we need to find $\alpha \in \mathbb{R}$ such that

$$\begin{aligned}
& \|\alpha y_v + \lambda x_v + (1 - \lambda)x_w\|^2 = \\
& = \alpha^2 \|y_v\|^2 + \|\lambda x_v + (1 - \lambda)x_w\|^2 + 2\alpha \langle y_v, \lambda x_v + (1 - \lambda)x_w \rangle = \\
& = \lambda \|x_v\|^2 + (1 - \lambda) \|x_w\|^2
\end{aligned} \tag{4.74}$$

By Corollary 2.15 of [Bauschke and Combettes \(2017b\)](#),

$$\|(\lambda x_v + (1 - \lambda)x_w)\|^2 = \lambda \|x_v\|^2 + (1 - \lambda) \|x_w\|^2 - \lambda(1 - \lambda) \|x_v - x_w\|^2. \tag{4.75}$$

Hence, (4.74) becomes

$$\alpha^2 \|y_v\|^2 + 2\alpha \langle y_v, \lambda x_v + (1 - \lambda)x_w \rangle - \lambda(1 - \lambda) \|x_v - x_w\|^2 = 0. \tag{4.76}$$

Note that by (4.76) it must be $\alpha \neq 0$ since x_v and x_w are distinct vectors. There exists α such that (4.73) is satisfied (i.e. $\tilde{x} \in \mathcal{S}(\lambda)$), if and only if

$$\Delta := (\langle y_v, (\lambda x_v + (1 - \lambda)x_w) \rangle)^2 + \lambda(1 - \lambda) \|y_v\|^2 \|x_v - x_w\|^2 \geq 0,$$

which holds for every $x_v, x_w \in H$. Then, there exists $\tilde{x} \in V \cap \mathcal{S}(\lambda)$ and the proof is finished by virtue of (4.67). $\square$

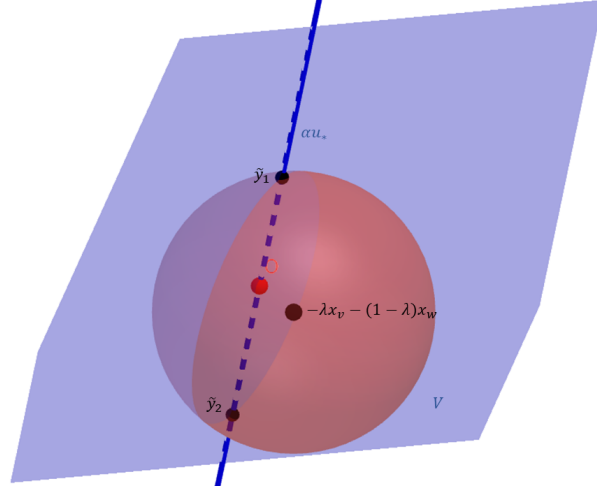


Figure 4.1: 3d representation of (4.66). The sphere $\mathcal{S}(\lambda)$ in red is centered in $-(\lambda(x_v) - (1 - \lambda)x_w)$. V is in light blue, while the line αy_v is in dark blue. $\mathcal{S}(\lambda)$ intersects αy_v in the points $\tilde{y}_1$ and $\tilde{y}_2$. 0 is in red. The picture was built with the 3D calculator of Geogebra, [Hohenwarter \(2002\)](#).

Remark 18. For $H = \mathbb{R}^n$, assumption *ii.* of Theorem 57 is automatically satisfied if $m + 1 < n$. For $H = \mathbb{R}^n$ and $m + 1 \leq n$, Theorem 57 can be considered a special case of Theorem 2.3 from [Beck \(2007\)](#).

Remark 19.

- i. In Figure 4.1, we provide a geometrical representation of the sets defined in (4.69), in the space of the variable $y \in \mathbb{R}^3$, with $m = 1$ such that $m + 1 = 2 < n = 3$. In $\mathbb{R}^3$, the above prove shows that the sphere $\mathcal{S}(\lambda)$ centered in $-(\lambda(x_v) + (1 - \lambda)x_w)$ of radius $\sqrt{\lambda\|x_v\|^2 + (1 - \lambda)\|x_w\|^2}$, always intersects the line αu_* of parameter α , where u_* belongs to a basis of the vector space V . In fact,

- the distance between 0 and $-(\lambda(x_v) + (1 - \lambda)x_w)$ is always less than the radius of $\mathcal{S}(\lambda)$ by (4.75),
 - the line αu_* passes through 0.
- ii. Observe that the assumption $\overline{m} < n$ is important for the validity of the geometric construction presented in the proof of Theorem 57 in finite dimensions. Otherwise, if $\overline{m} = n$, the only solution of the system (4.70) is $y = 0$ which implies that it must be $\tilde{x} = \lambda x_v + (1 - \lambda)x_w \in \mathcal{S}(\lambda)$, i.e. $\lambda x_v \perp (1 - \lambda)x_w$ which can hardly be satisfied.
- iii. Under the assumption of Theorem 57, there could exist an index $i \in \{0, \dots, m\}$ such that $f_i(x) := \|x\|^2$ and an index $j \in \{0, \dots, m\}$ such that $f_j(x) := -\|x\|^2$.

In this case, $\tilde{x}$ must satisfies

$$\begin{cases} \|\tilde{x}\|^2 \leq \lambda\|x_v\|^2 + (1 - \lambda)\|x_w\|^2 \\ -\|\tilde{x}\|^2 \leq -(\lambda\|x_v\|^2 + (1 - \lambda)\|x_w\|^2) \end{cases}.$$

The above proves that, in order to complete the proof of Theorem 57, we cannot choose $\tilde{x}$ such that

$$\|\tilde{x}\|^2 \neq \lambda\|x_v\|^2 + (1 - \lambda)\|x_w\|^2.$$

This motivates our approach of looking for suitable $\tilde{x}$ from among elements of $\mathcal{S}(\lambda)$.

Theorem 57 allows us to prove that (KKT) conditions are necessary and sufficient

optimality conditions for S-QCQP, as stated in the following theorem.

Theorem 58. (*Bednarczuk and Bruccola, 2023, Theorem 4.2*) *Let H be a Hilbert space. Consider S-QCQP such that Assumption 45 holds and let x^* be a global minimizer of S-QCQP. Moreover, suppose that one of the following holds:*

- i. H is infinite dimensional,*
- ii. H has n -dimensional, $n \in \mathbb{N}$. In this case, any maximally linearly independent subset of the system $\langle b_k, x \rangle = 0$, $k = 0, \dots, m$ has cardinality $\overline{m} < n$.*

Then the Fritz-John conditions (4.48) are necessary for optimality. Moreover, if there exists a point $x_0 \in H$ such that

$$f_k(x_0) < 0 \quad \forall k = 1, \dots, m,$$

then, for a feasible x^* , the conditions (KKT): there exists a vector $(\gamma_1, \dots, \gamma_m) \in \mathbb{R}_+^m \setminus 0$ such that

$$\begin{aligned} (i) \quad & 2a_J x^* + b_J + 2 \sum_{k=1}^m 2a_k x^* + b_k = 0, \\ (ii) \quad & \gamma_k f_k(x^*) = 0 \quad k = 1, \dots, m, \\ (iii) \quad & a_J + \sum_{k=1}^m \gamma_k a_k \geq 0, \end{aligned} \tag{KKT}$$

are necessary and sufficient for global optimality of x^* .

Proof. By Theorem 57, GIS is convex. We can apply Theorem 54 to complete the

proof. $A_J + \sum_{k=1}^m \gamma_k A_k$ can be rewritten as $(a_J + \sum_{k=1}^m \gamma_k a_k)I$ for S-QCQP. Then, condition (iii) of (KKT) becomes $a_J + \sum_{k=1}^m \gamma_k a_k \geq 0$ for S-QCQP. $\square$

Remark 20. Let $A : H \rightarrow H$ be a continuous self-adjoint linear operator. Moreover, let A be positive and bounded away from 0, i.e. there exists $\alpha > 0$ such that, $\forall x \in H, \langle x, Ax \rangle \geq \alpha \|x\|^2$, (see Pedersen, 2012, Proposition 3.2.6). Then the function $\|\cdot\|_A : H \rightarrow \mathbb{R}_+$ defined as $\|x\|_A = \sqrt{\langle x, Ax \rangle}$ is a norm in H (see Bonnans and Shapiro, 2013, section 3.3.2). With this observation, the characterization of Theorem 58 holds for scalar QCQP problems satisfying the following assumption.

Assumption 59. Let $A : H \rightarrow H$ be a continuous self-adjoint linear operator, positive and bounded away from zero. Let $A_J, A_k \forall k \in \{1, \dots, m\}$ be of the form $A_J = a_J A$, $A_k = a_k A$, where $a_J, a_k \in \mathbb{R}$. Then, A induces the norm $\|\cdot\|_A$.

Conditions for a positive continuous self-adjoint linear operator to be bounded away from zero can be found in Theorem 10.1 of Hestenes (1951).

Chapter 5

Solving the outer approximation subproblem at each step of Algorithm 1

5.1 Chapter Outline

- In Section 5.2, apply the results of Chapter 4, more specifically of section 4.7 to the outer approximation problem (OP_k), which is in general NP-hard.
- In Section 5.3, we show the connections between problem (OP_k) and the feasibility problem between a sphere and a polyhedron.
- In Section 5.4, we prove that the feasibility problem between a sphere and a polyhedron can be rewritten as a hidden convex QCQP, but it is still NP-hard.

5.2 Hidden convexity of the quadratic outer approximation problem under a dimension condition

The Algorithm (1) solves, at every iteration, the outer approximation problem (OP_k). Even if problem (OP_k) is a QCQP problem and QCQP problems are NP-hard, in

the recent years different algorithms based on branch-and-bound techniques were developed in order to find a global minimum of general QCQP problems, see Elloumi and Lambert (2019), Elloumi et al. (2024) and solvers such as Gurobi Optimization, LLC (2023) and Cplex (2009) have the option of solving a general QCQP.

In this section, we apply the results of Chapter 4, in particular Theorem 58, directly to problem (OP_k) . Consider the following condition.

Condition 60. Consider a (S-QCQP). Any maximally linearly independent subset of the system $\langle b_i, x \rangle = 0, i = 0, \dots, m$ has cardinality $\overline{m} < n$.

By Theorem 58, the following result holds.

Theorem 61. Consider a S-QCQP problem and assume that there exist some scalars $\gamma_i \geq 0$ not all null such that $a_0 + \sum_{i=1}^m \gamma_i a_i \geq 0$ and there exists a global minimizer x^* of S-QCQP. If there exists a point $x_0 \in \mathbb{R}^n$ such that

$$q_i(x_0) < 0 \quad \forall i = 1, \dots, m,$$

then, for a feasible x^* , the conditions KKT: there exists a vector $(\gamma_1, \dots, \gamma_m) \in \mathbb{R}_+^m \setminus 0$ such that

$$\begin{aligned} (i) \quad & 2a_0x^* + b_0 + 2 \sum_{i=1}^m \gamma_i a_i x^* + b_i = 0, \\ (ii) \quad & \gamma_i q_i(x^*) = 0 \quad i = 1, \dots, m, \\ (iii) \quad & a_0 + \sum_{i=1}^m \gamma_i a_i \geq 0, \end{aligned} \tag{KKT}$$

are sufficient for global optimality of x^* . Moreover, suppose that Condition 60 holds. Then, the (KKT) conditions are necessary and sufficient for global optimality of x^* .

Theorem 61 implies that S-QCQP is equivalent to its Lagrangian dual if Condition 60 holds, in the sense that strong duality holds (the optimal point of both problems is achieved and there is zero duality gap), see Section 4.5.

5.3 The feasibility problem between a sphere and a polyhedron

As pointed out in the previous subsection, the optimal value of problem (OP_k) , $\text{Opt}(OP_k)$, when Condition 60 holds, can be found by solving a convex program, e.g. the (SDP) relaxation. On the other hand, after finding the optimal value of (SDP), we still need to recover an optimal solution for (OP_k) . In fact, when $\text{Opt}(OP_k) = \text{Opt}(\text{SDP})$, we have

$$\{x \in \mathbb{R}^n \mid \text{Opt}(OP_k)\} \subseteq \{x \in \mathbb{R}^n \mid \text{Opt}(\text{SDP})\}. \quad (5.1)$$

Hence, in this section, we consider the case when we need to solve problem (OP_k) when $\text{Opt}(OP_k)$ is known and we define $\alpha = \text{Opt}(OP_k)$.

Without loss of generality, we may assume $J(x) = \|x\|^2$, i.e. $z = 0$.

We add to (OP_k) the redundant constraint $q_\alpha : \mathbb{R}^n \rightarrow (-\infty, +\infty]$, $q_\alpha(x) := \|x\|^2 - \alpha \geq 0$, which doesn't change the fact problem (OP_k) is a S-QCQP and doesn't impact Condition 60.

Notice that every quadratic constraint of the form $q_i(x) = -a_i\|x\|^2 + b_{i,k}^T x + c_k \leq 0$, $i = 1, \dots, m$, of (OP_k) , generated by Algorithm 1 intersects the closed ball $\overline{B(0, \alpha)}$.

Moreover, we can just remove the constraints which don't intersect the sphere $\{x \in \mathbb{R}^n \mid \|x\|^2 - \alpha = 0\}$, because the constraint $q_\alpha(x) \leq 0$, makes them redundant. The following result holds.

Lemma 62. Let $\text{Opt}(OP_k) = \alpha$. then solving the outer approximation problem (OP_k) , i.e.

$$\min_{x \in \mathbb{R}^n} J(x) \text{ s.t. } x \in \mathcal{O}_k, \quad (OP_k)$$

with

$$\mathcal{O}_k := \{x \in X \mid L_i(\varphi_i^k, x_k)(x) \leq 0, \quad i \in I(x_k)\} \cap \bigcap_{p=0}^{k-1} \{\mathcal{O}_p\},$$

is equivalent to solve the following auxiliary feasibility problem between a sphere and a convex set

$$\text{Find } x \in \mathcal{S}_k \cap \mathcal{O}_k^\alpha, \quad (5.2)$$

where

$$\begin{aligned} \mathcal{S}_k &:= \{x \in \mathbb{R}^n \mid \|x\|^2 = \alpha\} \\ \mathcal{O}_k^\alpha &:= \{x \in \mathbb{R}^n \mid b_{i,k}^T x \leq a_i \alpha - c_{i,k}, \quad i \in I(x_k)\} \cap \bigcap_{t=0}^{k-1} \{\mathcal{O}_t^\alpha\}. \end{aligned}$$

Proof. Since $\text{Opt}(OP_k) = \alpha$, then every optimal solution x_{k+1} of problem (OP_k) satisfies $\alpha = \|x_{k+1}\|^2$ and, for $i \in I(x_k)$,

$$0 \geq L_i(\varphi_i^k, x_k)(x_{k+1}) = b_{i,k}^T x_{k+1} + a_i \|x_{k+1}\|^2 + c_{i,k} = b_{i,k}^T x_{k+1} + a_i \alpha + c_{i,k}.$$

Hence, every global minimum of problem (OP_k) belongs also to $x \in \mathcal{S}_k \cap \mathcal{O}_k^\alpha$. Since

every point $x \in \mathcal{S}_k$ satisfies $J(x) = \alpha$, then $\mathcal{S}_k \cap \mathcal{O}_k^\alpha$ is the set of global minima for problem (OP_k) . $\square$

5.3.1 Nonzero recession cone of the polyhedron $\mathcal{O}_k^\alpha$

In case that the polyhedron $\mathcal{O}_k^\alpha$ contains a half-line, it is possible to find the intersection between $\mathcal{O}_k^\alpha$ and the sphere $\mathcal{S}_k$, even when (P) is high dimensional. Consider the following definition of recession cone.

Definition 63. (Rockafellar, 1997, Section 8) Let C be a non-empty convex set in $\mathbb{R}^n$. Then, C recedes in the direction of $y \neq 0$ if and only if $x + \lambda y \in C$ for all $\lambda \geq 0$ and $x \in C$. The set of all vectors $y \in \mathbb{R}^n$ satisfying the latter condition, including $y = 0$, is called recession cone of C , which we denote $\text{rec } C$.

Notice that we always have $0 \in \text{rec } C$ and $C \subset \mathbb{R}^n$ is bounded if and only if $\text{rec } C = \{0\}$ (Bauschke and Combettes, 2017b, Proposition 6.49 and Corollary 6.52).

Since $\mathcal{O}_k^\alpha$ is a polyhedral set, we can represent it as $\{x \in \mathbb{R}^n \mid A_\alpha x \leq b_\alpha\}$ for $A_\alpha \in \mathbb{R}^{m \times n}$ and $b_\alpha \in \mathbb{R}^m$. Then $\text{rec } C = \{x \in \mathbb{R}^n \mid A_\alpha x \leq 0\}$, see (Rockafellar, 1997, Section 8). The following remark holds.

Remark 21. By Definition 63, if we find $y \in \text{rec } C$, $y \neq 0$, the half-line passing between y and 0, is a direction of recession. Given any point $x \in \mathcal{O}_k^\alpha$, we can scale x along the direction $[0, y]$ in order to find a new point $x_0 \in \mathcal{O}_k^\alpha \cap \mathcal{S}_k$ (see also (Rockafellar, 1997, Theorem 19.5)).

We point out that when Condition 60 is satisfied, by the rank nullity theorem applied to $A_\alpha x = b_\alpha$, the polyhedron $\mathcal{O}_k^\alpha$ contains at least one half-line. On the other

hand, when $\mathcal{O}_k^\alpha$ doesn't contain half-lines (i.e. $\mathcal{O}_k^\alpha$ is a polytope) and Condition 60 doesn't hold we don't know what is $Opt(OP_k)$ *a priori* (while it is clear that if $\mathcal{O}_k^\alpha$ contains a halflines, then it intersects the sphere $\mathcal{S}_k$ and thus $Opt(OP_k) = J(x_k) = \alpha$). Then, we can take $\alpha = \|x_k\|^2$. If $\mathcal{S}_k \cap \mathcal{O}_k^\alpha = \emptyset$, it means that the outer approximation constraints set $\mathcal{O}_k$ covers $\text{lev}_{\leq \alpha} J$.

5.3.2 The case when $\mathcal{O}_k^\alpha$ is a polytope

In case $\mathcal{O}_k^\alpha$ is a polytope (hence $\text{rec } \mathcal{O}_k^\alpha = \{0\}$), in order to find $x \in \mathcal{S}_k \cap \mathcal{O}_k^\alpha$, if exists, it suffices to find two points: $x_1 \in \mathcal{O}_k^\alpha \setminus B(0, \sqrt{\alpha})$ and the point of $\mathcal{O}_k^\alpha$ with minimal norm x_2 (the projection of 0 onto $\mathcal{O}_k^\alpha$). After that, the $x \in \mathcal{S}_k \cap \mathcal{O}_k^\alpha$ will be in the segment connecting x_1 and x_2 by convexity of $\mathcal{O}_k^\alpha$.

Notice that, if a point x_1 doesn't exist, than problem (5.2) has no solutions. Geometrically, this means that the quadratic constraints of $\mathcal{O}_k$ cut out the level set $\{x \in \mathbb{R}^n \mid J(x) = \|x\|^2 \leq \alpha\}$, i.e. $\{x \in \mathbb{R}^n \mid J(x) = \|x\|^2 \leq \alpha\} \cap \mathcal{A} = \emptyset$, where $\mathcal{A}$ is the constraint set of (P). This situation is considered in the next section.

In order to find x_1 , we maximize the Euclidean norm over the polyhedron $\mathcal{O}_k^\alpha$, which is a NP-hard problem in general (see e.g. Pardalos and Rosen (1986)). On the other hand, at this stage, and we are only focusing in the case where $\mathcal{O}_k^\alpha$ is a polytope, which is a polyhedron that doesn't contains halflines.

Since the objective function is concave, i.e. $-\|x\|^2$, the global solution of (OP_k) occur in a vertex of the polyhedron $\mathcal{O}_k^\alpha$.

By Khachiyan et al. (2009) it isn't proved that generating all vertices of a polytope is NP-hard, i.e. we don't know if the complexity grows exponentially with respect

to the dimension of the space $\mathbb{R}^n$. In case of a simple polytope (or nondegenerate, i.e. at every vertex v exactly n constraints are active), the problem is not NP-hard (the complexity grows polynomially with respect to the dimension of the space). Moreover, by [Chazelle \(1993\)](#), for fixed dimensions, the complexity doesn't increase exponentially, i.e. with increasing number of constraints, we are able to find all the vertices of a polytope efficiently (in polynomial time).

We summarize the discussion above with the following remark.

Remark 22. We say that the feasibility problem (5.2) is solved if we find a global solution of (5.2) or if we establish that (5.2) has no solutions. When $\mathcal{O}_k^\alpha$ is a polytope, solving the feasibility problem (5.2) is equivalent to find all the vertices of $\mathcal{O}_k^\alpha$. For fixed dimensions, there exists algorithms that solves (5.2) in polynomial time (in case $\mathcal{O}_k^\alpha$ isn't a polytope, we don't have to calculate all the vertices, but we can find a solution of problem (5.2) as discussed in Remark 21).

5.4 Hidden convex problems can be NP-hard

We show that (OP_k) can be hidden convex even if Condition 60 doesn't hold.

Notice that (5.2) can be rewritten as

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \|x\|^2 \\ \text{s.t. } \quad & x \in \mathcal{O}_k^\alpha, \quad \|x\|^2 \leq \alpha \end{aligned} \tag{5.3}$$

Clearly, Problem 5.3 is a special case of problem (S-QCQP), hence Theorem 58 holds. Moreover, the following theorem shows that Problem 5.3 is hidden convex

when the optimal values $\bar{J}$ satisfies $\bar{J} = \alpha$, even if Condition 60 doesn't hold.

Theorem 64. *Consider Problem 5.3. Let $\bar{J}$ be the optimal value of Problem 5.3, with $\bar{J} = \alpha$. Under the Slater constraint qualification ($\exists x_0 \in \mathbb{R}^n$ such that $x_0 \in \text{int } \mathcal{O}_k^\alpha \cap \{x \in \mathbb{R}^n \mid \|x\|^2 - \alpha \leq 0\}$), the conditions (KKT) are necessary and sufficient for global optimality.*

Proof. The sufficiency part is well known under the Slater condition, so we focus on the necessity part. Let $\bar{x}$ be a global minimizer of Problem 5.3, with $J(\bar{x}) = \bar{J} = \alpha$. Take $\gamma = (1, 0, \dots, 0) \in \mathbb{R}^{m+1}$, where m is the number of linear constraints of $\mathcal{O}_k^\alpha$ and $\gamma_1 = 1$ is the Lagrangian multiplier associated with the constraint $\|x\|^2 - \alpha \leq 0$. Condition (iii) of (KKT) holds, in fact

$$a_0 + \sum_{i=1}^m \gamma_i a_i = -1 + \gamma_1 = 0.$$

Since $\|\bar{x}\|^2 - \alpha = 0$, we have $\gamma_1(\|\bar{x}\|^2 - \alpha) = 0$. Condition (ii) of (KKT) holds. The Lagrangian function of Problem 5.3, for $\gamma_i = 0$, $i = 2, \dots, m$ is

$$L(x, \gamma_1) := -\|x\|^2 + 2\gamma_1\|x\|^2$$

Condition (i) of (KKT), in the case of Problem 5.3, with $\gamma_i = 0$, $i = 2, \dots, m+1$, reads as

$$\nabla_x L(x, \gamma_1) = 0 \Leftrightarrow -2x_i + 2\gamma_1 x_i = 0, \quad i = 1, \dots, n. \quad (5.4)$$

□

Remark 23. By Theorem 47, under the assumptions of Theorem 64, Problem 5.3 is

hidden convex when $\bar{J} = \alpha$ and its optimal value and the optimal value of its SDP relaxation are the same.

The next result prove that Problem 5.3 is NP-hard even when $\bar{J} = \alpha$, and hence hidden convexity doesn't exclude the NP-hardness.

Proposition 65. Consider Problem 5.3. Let $\bar{J}$ be the optimal value of Problem 5.3, with $\bar{J} = \alpha$ and assume that the Slater condition holds. Assume that Condition 60 doesn't hold. Then the following holds:

- i. Problem 5.3 in $\mathbb{R}^n$ is hidden convex and its optimal value and the value of its SDP relaxation are the same. However, recovering an optimal feasible solution for Problem 5.3 is NP-hard.
- ii. The KKT conditions (KKT) are necessary and sufficient for global optimality. However, solving the KKT conditions is in general NP-hard.

Proof. i. In view of (5.1), the only information that we get by solving the SDP relaxation is the optimal value of Problem 5.3. Hence, in order to recover an optimal feasible solution for Problem 5.3 we need to find the intersection between the constraint set $\mathcal{O}_k^\alpha$ and the sphere $\{x \in \mathbb{R}^n \mid J(x) = \|x\|^2 = \bar{J}\}$. In view of Lemma 62, the feasibility problem between the constraint set $\mathcal{O}_k^\alpha$ and the sphere $\{x \in \mathbb{R}^n \mid J(x) = \|x\|^2 = \alpha\}$ is equivalent to Problem 5.3. Moreover, this problem is NP-hard, see section 5.3.

- ii. Since $\bar{J} = \alpha$, the constraints $\|x\|^2 \leq \alpha$ is active in every point $\bar{x}$ which satisfies (KKT). By (iii) of (KKT), the dual variable $\gamma_0 \geq 0$ related to $\|x\|^2 \leq \alpha$ must

satisfy $\gamma_0 \geq 1$. Let γ_i , $i = 1, \dots, m$ be the dual variable related to the active m linear constraints of $\mathcal{O}_k^\alpha$. If $\gamma_0 = 1$, the (KKT) conditions are not useful for solving Problem 5.3. In fact, by stationarity ((i) of (KKT)), we would have

$$-2\bar{x} + 2\gamma_0\bar{x} + \sum_{i=1}^m \gamma_i b_i = 0 \Rightarrow \sum_{i=1}^m \gamma_i b_i = 0.$$

Hence, stationarity doesn't relate the dual variable γ with $\bar{x}$ and the (KKT) conditions don't provide with any non trivial characterization of the global minimum of Problem 5.3, when $\gamma_0 = 1$. We show in the following, that it is possible that all $(\gamma_0, \gamma_1, \dots, \gamma_m)$ which satisfies (KKT) at a global minimum $\bar{x}$ may have $\gamma_0 = 1$.

Take $\gamma_0 > 1$. By stationarity we have

$$-2\bar{x} + 2\gamma_0\bar{x} + \sum_{i=1}^m \gamma_i b_i = 0 \Rightarrow \frac{\sum_{i=1}^m \gamma_i b_i}{2(1 - \gamma_0)} = \bar{x}.$$

By the complementarity conditions ((ii) of (KKT)) for the active linear constraints $b_i^T x + c_i \leq 0$, $i = 1, \dots, m$, the dual variables $\gamma_0, \gamma_1, \dots, \gamma_m$ must satisfy

$$\left\{ \begin{array}{l} (\sum_{i=1}^m \gamma_i b_i)^T b_1 = \gamma_1 \|b_1\|^2 + \sum_{i \neq 1} \gamma_i b_i^T b_1 = -2c_1(1 - \gamma_0), \\ \dots, \\ (\sum_{i=1}^m \gamma_i b_i)^T b_m = \gamma_m \|b_m\|^2 + \sum_{i \neq m} \gamma_i b_i^T b_m = -2c_m(1 - \gamma_0). \end{array} \right. \quad (5.5)$$

Notice that (5.5) is a linear system with m equation and $m+1$ variables $(\gamma_0, \gamma_1, \dots, \gamma_m)$,

but we should not forget about the constraint $\gamma_0 > 1$. By (5.5), the following holds

$$-2 \sum_{i=1}^m \gamma_i c_i (1 - \gamma_0) = \sum_{i=1}^m \gamma_i^2 \|b_i\|^2 + \sum_{i=1}^m \sum_{j \neq i}^m \gamma_i \gamma_j b_j^T b_i. \quad (5.6)$$

Now consider the complementarity condition for the quadratic constraint of Problem 5.3.

We have

$$\begin{aligned} \|\bar{x}\|^2 &= \alpha \\ \left\| \sum_{i=1}^m \gamma_i b_i \right\|^2 &= \alpha 4(1 - \gamma_0)^2 \\ \sum_{i=1}^m \gamma_i^2 \|b_i\|^2 + \sum_{i=1}^m \sum_{j \neq i}^m \gamma_i \gamma_j b_i^T b_j &= \alpha 4(1 - \gamma_0)^2 \\ -2 \sum_{i=1}^m \gamma_i c_i (1 - \gamma_0) &= \alpha 4(1 - \gamma_0)^2 \\ \sum_{i=1}^m \gamma_i c_i &= -\alpha 2(1 - \gamma_0) = 2\alpha(\gamma_0 - 1), \end{aligned} \quad (5.7)$$

where the fourth line holds thanks to (5.6). For example, let $\bar{x}$ be a global solution of Problem 5.3 with $\|\bar{x}\|^2 = \alpha$, which means that the intersection of the sphere and the polyhedral defined in (5.2) isn't empty. Let $b_1^T \bar{x} + c_1 = 0$ be the only linear constraint active at $\bar{x}$. Then we must have $c_1 > 0$, in fact by (5.7)

$$\gamma_1 = 2\alpha(\gamma_0 - 1)/c_1 \geq 0 \Rightarrow x = \gamma_1 b_1 / 2(1 - \gamma_0) = -\alpha b_1 / c_1.$$

Hence, if instead $c_1 \leq 0$ and $b_1 \neq 0$, only $(\gamma_0, \gamma_1, \dots, \gamma_m) = (1, 0, \dots, 0)$ satisfies

(KKT) at $\bar{x}$.

□

Chapter 6

Computational realizations of the cutting spheres method

6.1 Chapter Outline

- In Section 6.2, we describe the Cutting sphere algorithm with warm restarts. With respect to Algorithm 1, the Cutting sphere algorithm with warm restarts has less strong convergence properties, but it is more suitable for practical applications. In particular, the number of constraints of the outer approximation problems solved at each iteration by the Cutting sphere algorithm with warm restarts is always bounded. The Cutting sphere algorithm with warm restarts always stops in a finite number of steps when its parameter δ , satisfies $\delta > 0$, see Proposition 66. In this case, the outcome of the Cutting sphere algorithm with warm restarts is studied in subsection 6.2.2. When $\delta = 0$ and the Cutting sphere algorithm with warm restarts doesn't terminate in finite number of steps, the convergence result is studied in section 6.2.3.

- In Section 6.3, we present the inexact cutting sphere algorithm, which is a modification of the Cutting sphere algorithm with warm restarts. It exploits the result of Chapter 5 and guarantees that the complexity bound for solving an outer approximation step doesn't increase exponentially with the number of constraints.

6.2 Cutting Sphere with Warm Restart

6.2.1 A warm restart algorithm

The number of constraints m_k of the set $\mathcal{O}_k$ in Algorithm 1 becomes increasingly large as the iteration number k grows. To overcome this drawback, we introduce a warm restart procedure in the proposed outer approximation approach as detailed in Algorithm 2.

In contrast to Algorithm 1, Algorithm 2 can perform two types of iterations, namely *restart iterations* and *cumulative iterations*. At iteration number $k \in \mathbb{N}$, the following notation will be used:

- r_k is the index of the last *restart iteration*,
- $\bar{m} > 0$ is a positive real number, which is used as an upper bound on the computational cost m_k of computing a cumulative iteration at k (m_k can be, for example, the number of constraints that $\mathcal{O}_k$ would have if k is a cumulative iteration. So, m_k can be calculated before deciding if k will be a cumulative iteration or not).

A *cumulative iterations* keep all the constraints, which were built in the algorithm in

the previous steps since the last *restart iteration*. $\mathcal{O}_k$ can still be built recursively as in (3.13). The difference, however, with the cutting sphere algorithm 1 is that we only aggregate constraints of the form $L_i(\varphi_i^\ell, x_\ell)(x) \leq 0$ for $i \in I(x_\ell)$ and $\ell \in \{r_k, \dots, k\}$.

A *restart iteration* is performed whenever the following two conditions are simultaneously met:

- the value of the current iteration, $J(x_k)$, strictly increases with respect to the last *restart iteration*, i.e., $J(x_k) > J(x_{r_k})$,
- the computing cost m_k of a cumulative iteration step at k exceeds the chosen bound $\overline{m}$.

Unlike *cumulative iterations*, a *restart iteration* takes into account only the constraints built at the current iteration and the lower level bound $J(x_k)$, i.e., the approximation set is the given by

$$\mathcal{O}_k = \{x \in \mathbb{R}^n \mid (\forall i \in I(x_k)) L_i(\varphi_i^k, x_k)(x) \leq 0 \text{ and } J(x) \geq J(x_k)\}. \quad (6.1)$$

A restart iteration is always followed by a cumulative one since the condition $J(x_{k+1}) = J(x_{r_{k+1}})$ is then satisfied in Algorithm 2. It is worth observing that, because of the quadratic form of function J , solving the problem $\min_{x \in \mathbb{R}^n} J(x)$ s.t. $x \in \mathcal{O}_k$ remains a S-QCQP problem as expressed by (4.1).

Algorithm 2 Cutting spheres with warm restarts

Ensure: $x_0 = z$, $r_0 = 0$, $k = 0$, $\mathcal{O}_{-1} = \mathbb{R}^n$, $m_0 = 0$, $s_{-1} = 0$, $\delta \geq 0$, $\bar{m} > 0$

```

1: while  $s_{k-1} = 0$  do
2:    $s_k = 1$ 
3:   if  $(\exists i \in I) f_i(x_k) > 0$  then
4:     if  $m_k \geq \bar{m}$  then
5:       if  $J(x_k) > J(x_{r_k}) + \delta$  then
6:          $r_{k+1} = k + 1$ 
7:         Build  $\mathcal{O}_k$  as in (6.1)      % restart iteration
8:          $s_k = 0$ 
9:       end if
10:    else
11:       $r_{k+1} = r_k$ 
12:      Build  $\mathcal{O}_k$  as in (3.13)      % cumulative iteration.
13:       $s_k = 0$ 
14:    end if
15:    if  $s_k = 0$  then
16:       $x_{k+1} \in \arg \min_{x \in \mathcal{O}_k} \|x - z\|^2$       % Solution of outer approximation
        problem
17:    end if
18:  end if
19:   $k \leftarrow k + 1$ .
20: end while

```

Algorithm 2 generates a sequence $(x_k)_{k \in \mathbb{L}}$ where $\mathbb{L} \subseteq \mathbb{N}$ is the set of running iterations. Let $\mathbb{K}$ be the set of restart iterations, defined as

$$\mathbb{K} = \{k \in \mathbb{L} \setminus \{0\} \mid r_k \neq r_{k-1}\}. \quad (6.2)$$

So basically $\mathbb{K}$ is the set of iteration numbers for which line 6 in Algorithm 2 is performed. We are introducing $\mathbb{L} \subset \mathbb{N}$, instead of just writing $k \in \mathbb{N}$ because Algorithm 2 may stop in a finite number of steps (always for $\delta > 0$, as shown in the next proposition).

Let $\mathcal{S}_{\mathcal{A}}$ be solution set of problem (P). We have the following properties of Algorithm 2.

Proposition 66. Under the assumptions of Theorem 30, consider Algorithm 2. Let $(\mathcal{O}_{k-1}, x_k)_{k \in \mathbb{L}}$ be generated by Algorithm 2.

- i. For every $k \in \mathbb{L}$, we have $\mathcal{O}_{k-1} \subseteq \mathcal{O}_{k-2} \subseteq \dots \subseteq \mathcal{O}_{r_k-1}$.
- ii. For every $k \in \mathbb{L}$, $\mathcal{A} \subseteq \mathcal{O}_{k-1}$.
- iii. $(J(x_k))_{k \in \mathbb{L}}$ is a nondecreasing sequence bounded from above by J^* and $(J(x_k))_{k \in \mathbb{K}}$ is an increasing sequence.
- iv. For every $k \in \mathbb{L}$, $I(x_k) = \emptyset$ if and only if $x_k \in \mathcal{S}_{\mathcal{A}}$.
- v. If $\delta > 0$, then
 - (a) for every $k \in \mathbb{L}$, the number of constraints in $\mathcal{O}_{k-1}$ never exceeds $\overline{m}$;

- (b) there exists $k^* \in \mathbb{N}$ such that $s_{k^*} = 1$ (so, the algorithm “stops in a finite number of steps”);
- (c) either $x_{k^*} \in \mathcal{S}_A$, or the resulting lower approximation to J^* is $J(x_{k^*})$.

Proof.

- i. These inclusions follow from the fact that, between two consecutive restarts, Algorithm 2 accumulates constraints.
- ii. Let us show the property by induction. It is obviously satisfied when $k = 0$. Assume that $k + 1 \in \mathbb{L}$ and that the property is satisfied at index k and let $x \in \mathcal{A}$. Two cases may arise. If $\mathcal{O}_k$ is built according to (3.13), since $x \in \mathcal{O}_{k-1}$ and

$$x \in \mathcal{A} \subseteq \{x \in \mathbb{R}^n \mid (\forall i \in I(x_k)) L_i(\varphi_i^k, x_k)(x) \leq 0\}, \quad (6.3)$$

we deduce that $x \in \mathcal{O}_k$. When $\mathcal{O}_k$ is built by (6.1), since x_k minimizes J on $\mathcal{O}_{k-1}$ and $x \in \mathcal{O}_{k-1}$, $J(x) \geq J(x_k)$. Combining this inequality with (6.3) and (6.1) shows that $x \in \mathcal{O}_k$.

- iii. In a cumulative step, the non-decreasing property follows from (i). In turn, the last inequality constraint in (6.1) ensures that $J(x_{k+1}) \geq J(x_k)$ in a restart iteration. The fact that, for every $k \in \mathbb{L}$, $J(x_k) \leq J^*$ is a consequence of (ii). Let $(k_1, k_2) \in \mathbb{K}^2$ be such that $k_1 = \max\{k \in \mathbb{K} \mid k < k_2\}$. Then $J(x_{k_2}) > J(x_{k_1}) + \delta$. This shows that $(J(x_k))_{k \in \mathbb{K}}$ is an increasing sequence with jumps greater than δ .
- iv. $I(x_k) = \emptyset \Leftrightarrow x_k \in \mathcal{A}$. Since $J(x_k) \leq J^*$, this is also equivalent to $x_k \in \mathcal{S}_A$.

- v. (a) This follows from the construction of Algorithm 2.
- (b) $s_k = 1$ is a stopping criterion as we repeat the while loop as long as $s_{k-1} = 0$.

Two situation may appear when running Algorithm 2.

1. There exist $k^* \in \mathbb{L}$ such that $I(x_{k^*}) = \emptyset$.
2. For any $k \in \mathbb{L}$, $I(x_k) \neq \emptyset$.

In case 1., by (iv), $x_{k^*} \in \mathcal{S}_A$ and $s_{k^*} = 1$. Let us show that, in case 2., the algorithm also terminates in a finite number of steps.

We will show that the number of restart iterations is finite.

By (iii), for every $(k_1, k_2) \in \mathbb{K}^2$ with $k_2 > k_1$,

$$J(x_{k_2}) > J(x_{k_1}) + \delta$$

We deduce that, for every $n \in \mathbb{N}$, $(k_1, k_2, \dots, k_n) \in \mathbb{K}^n$ with $k_n > k_{n-1} > \dots > k_1$, we have

$$J(x_{k_n}) \geq J(x_{k_1}) + (n - 1)\delta.$$

However, $J(x_{k_n}) \leq J^* < +\infty$. Therefore, the set $\mathbb{K}$ must be finite and the following holds:

$$(\exists \bar{k} \in \mathbb{K})(\forall k \in \mathbb{L} \cap [\bar{k}, +\infty[) r_k = \bar{k}. \quad (6.4)$$

For every $k \in \mathbb{L} \cap [\bar{k}, +\infty[$ the set $\mathcal{O}_k$ can be built only as in (3.13) and the number of constraints m_k will be increasing for $k \geq \bar{k}$. Hence, there exists

$k^* \geq \bar{k}$ such that $m_{k^*} \geq \bar{m}$. As no restart occurs, $J(x_{k^*}) \leq J(x_{r_{k^*}}) + \epsilon = J(x_{\bar{k}}) + \epsilon$, so Algorithm 2 stop at iteration $k^* + 1$, since $s_{k^*} = 1$.

(c) The proof follows from (iii) and (iv).

□

6.2.2 Outcome of the Cutting spheres Algorithm with warm restarts 2

According to Proposition 66, for any $\delta > 0$, Algorithm 2 stops in a finite number of k steps with either (a) x_k being feasible or (b) x_k being infeasible. Case (a) implies that the original problem (P) is solvable in a finite number of steps (by arguments analogous to those in Lemma 28).

On the other hand, in case (b), a natural and interesting question is how far from the feasibility or equivalently, how far from the optimality is the outcome x_k . The answer to this question could be provided by investigating the so-called *global error bound* property as explained below.

We say that $\mathcal{A}$ (the feasible set of (P)) enjoys the *global error bound property* if there exists a constant $\tau > 0$ such that

$$(\forall z \in \mathbb{R}^n) \quad d(z, \mathcal{A}) \leq \tau \max_{1 \leq i \leq m} (f_i(z))_+, \quad (6.5)$$

where $d(z, \mathcal{A}) = \inf_{a \in \mathcal{A}} \|z - a\|$ and, for every $i \in \{1, \dots, m\}$, $(f_i(z))_+ := \max\{0, f_i(z)\}$.

A lot of research has been devoted to necessary/sufficient conditions ensuring (6.5) in the case when $m = 1$ and f_1 is convex, see e.g. Cuong and Kruger (2022); Fabian

et al. (2010). For $m > 1$ and $(f_i)_{1 \leq i \leq m}$ convex, sufficient conditions can be found in Lewis and Pang (1998).

It is not our aim here to provide a detailed analysis of the error bound inequality (6.5) when, for every $i \in I$, f_i is ρ_i -weakly convex. Instead, we observe that sufficient (and necessary) conditions ensuring the error bound inequality for convex functions $(f_i)_{1 \leq i \leq m}$ almost straightforwardly lead to conditions ensuring the error bound property for weakly convex functions.

For a given $i \in \{1, \dots, m\}$, f_i is proper, lsc, and ρ_i -weakly convex with $\rho_i \in \mathbb{R}_+$. Then, the function $\tilde{f}_i$ defined as

$$\tilde{f}_i := f_i + \frac{\rho_i}{2} \|\cdot\|^2,$$

is proper, lsc, and convex with domain E . Let

$$\mathcal{A}^c = \{x \in \mathbb{R}^n \mid (\forall i \in I) \tilde{f}_i(x) \leq 0\}.$$

Clearly,

$$\mathcal{A}^c \subset \mathcal{A} \subset E. \tag{6.6}$$

Hence, by Proposition 8 of Lewis and Pang (1998), we obtain the following fact.

Proposition 67. Assume that

$$\text{int } \mathcal{A}^c = \{x \in \mathbb{R}^n \mid (\forall i \in \{1, \dots, m\}) \tilde{f}_i(x) < 0\} \neq \emptyset$$

and there exists a scalar $\kappa > 0$ such that

$$(\forall \bar{x} \in \text{bd } \mathcal{A}^c) \quad (\exists \bar{a} \in \text{int } \mathcal{A}^c) \quad \frac{\|\bar{x} - \bar{a}\|}{\min_{1 \leq i \leq m} -\tilde{f}_i(\bar{a})} \leq \kappa. \quad (6.7)$$

Then there exists a constant $\tau > 0$ such that

$$(\forall x \in \mathbb{R}^n) \quad \text{dist}(x, \mathcal{A}) \leq \text{dist}(x, \mathcal{A}^c) \leq \tau \max_{1 \leq i \leq m} (\tilde{f}_i(x))_+. \quad (6.8)$$

As a consequence of this result, if $\text{int } \mathcal{A}^c \neq \emptyset$, (6.7) holds, and $f_i(x_k) > 0$ for every $i \in I(x_k)$ then, by (6.8),

$$\text{dist}(x_k, \mathcal{A}) \leq \tau \max_{i \in I(x_k)} (f_i(x_k) + \frac{\rho_i}{2} \|x_k\|^2),$$

6.2.3 Case $\delta = 0$

In this section we consider Algorithm 2 in the case when δ is zero. We introduce the following definition.

Definition 68. We say that a subsequence $(x_p)_{p \in \mathbb{N}}$ produced by Algorithm 2 with $\delta = 0$ is a *non cutting subsequence* (NCS) if there exists $i \in I$ such that

$$\exists \bar{p} \in \mathbb{N} \text{ such that } (\forall r > p > \bar{p}) \quad L_i(\varphi_p^i, x_p)(x_r) > 0. \quad (\text{NCS})$$

Remark 24. By construction, if a subsequence $(x_p)_{p \in \mathbb{N}}$ produced by Algorithm 2 is

(NCS), then there exists $\bar{p} \in \mathbb{N}$ such that $(\forall p > \bar{p}), r_p < r_{p+1}$, i.e. at least one restart iteration was performed between every iteration of the (NCS) subsequence.

Two cases can appear.

- The cardinality of $\mathbb{K}$, defined by (6.2), is finite, i.e. $|\mathbb{K}| \in \mathbb{N}$. By Remark 24, there doesn't exist any (NCS) subsequence.
- $\mathbb{K}$ has an infinite number of elements, i.e. there exists a infinite subsequence $\{x_{k_m}\}_{m \in \mathbb{N}}$, where each x_{k_m} , $m \in \mathbb{N}$, is a restart.

The following theorem analyzes the behaviour of Algorithm 2 in case $\delta = 0$.

Theorem 69. *Under the assumptions of Theorem 30, consider Algorithm 2 with $\delta = 0$. Let $(x_k)_{k \in \mathbb{N}}$ be the sequence produced by Algorithm 2. Points (i)-(iv) of Proposition 66 holds.*

Assume that Algorithm 2 does not terminates in a finite number of steps. Then the cardinality of $\mathbb{K}$, defined in (6.2), is not finite. Moreover, $(x_k)_{k \in \mathbb{N}}$ possesses a convergent subsequence and every convergent subsequence of $(x_k)_{k \in \mathbb{N}}$ which does not satisfy (NCS) converges to a global minimum of (P).

Proof. If $(x_k)_{k \in \mathbb{N}}$ is an infinite subsequence, the stopping condition $s_k = 1$ is never met. This means that the number of restarts must be infinite, i.e. $|\mathbb{K}| = \infty$. Otherwise, there exists $\bar{k} \in \mathbb{K}$ such that for all $k \geq \bar{k}$, we perform cumulative iterations only. Then, the number of constraints of the outer approximation problem increases as k grows. The m_k grows, and the condition $m_k \geq \bar{m}$ will be met at some iteration $k \geq \bar{k}$, while the condition at step 5, $J(x_k) > J(x_{r_k})$ will not be satisfied (by assumption). Then, the stopping condition will be active at iteration k , i.e. $s_k = 1$.

Let $(x_{k_l})_{l \in \mathbb{N}}$ be a convergent subsequence of $(x_k)_{k \in \mathbb{N}}$ which does not satisfy (NCS). Since the number of $i \in I$ is finite, without loss of generality we may consider the following subcases for a fixed $i \in I$:

(1): $f_i(x_{k_l}) \leq 0$ for infinite number of l for all $i \in I$.

(2): $f_i(x_{k_l}) > 0$ for all $l \geq \bar{l} \in \mathbb{N}$ (note that $\bar{l}$ can be independent of i).

Fix $i \in I$.

If case (1) holds, then by the lower semicontinuity of f_i , we obtain that $f_i(\bar{x}) \leq 0$.

If case (2) holds, since $(x_{k_l})_{l \in \mathbb{N}}$ does not satisfy (NCS), then there exists a sequence l_n such that

$$L_i(\varphi_i^{k_{l_n}}, x_{k_{l_n}})(x_{k_{l_n+1}}) = f_i(x_{k_{l_n}}) + \varphi_i^{k_{l_n}}(x_{k_{l_n+1}}) - \varphi_i^{k_{l_n}}(x_{k_{l_n}}) \leq 0, \quad \forall n \in \mathbb{N}. \quad (6.9)$$

By Theorem 29 (1), for $k_{l_n} \in \mathbb{N}$, there exists a subsequence $k_{l_{nm}} \in \mathbb{N}$, such that $\{\varphi_i^{k_{l_{nm}}}\}$ converges to $\bar{\varphi}_i \in \partial_{lsc} f_i(\bar{x})$. Without loss of generality, we may assume that $k_{l_{nm}}, m \in \mathbb{N}$ is the same as $k_{l_n}, n \in \mathbb{N}$.

By repeating the calculations in the proof of Theorem 30, for $l_n \rightarrow +\infty$,

$$\varphi_i^{k_{l_n}}(x_{k_{l_n+1}}) - \varphi_i^{k_{l_n}}(x_{k_{l_n}}) \rightarrow 0.$$

Hence, by lower semicontinuity, for $x_{k_{l_n}} \rightarrow \bar{x}$, we have $f_i(\bar{x}) \leq 0$. This reasoning holds for all $i \in I$ and hence $\bar{x} \in \mathcal{A}$ and it is a global minimum for (P). $\square$

Notice that we do not know if the limit of a (NCS) convergent subsequence is a global minimum of problem (P) or not. The following corollary of Theorem 69,

however, present some additional results on converging (NCS) subsequences.

Corollary 70. Under the assumptions of Theorem 69, let $(x_k)_{k \in \mathbb{N}}$ be a sequence produced by Algorithm 2. A convergent (NCS) subsequence $(x_p)_{p \in \mathbb{P} \subset \mathbb{N}}$ of $(x_k)_{k \in \mathbb{N}}$ converges to the global minimum of problem (P) if there exists a convergent supersequence of $(x_p)_{p \in \mathbb{P}}$ which is not (NCS).

Moreover, if there exists a convergent subsequence which is not (NCS), every convergent subsequence of $(x_k)_{k \in \mathbb{N}}$ converges to a point $\bar{x} \in \mathbb{R}^n$ which satisfies $J(\bar{x}) = J^*$, where J^* is the optimal value of problem (P).

Proof. The first part of the proof is trivial. In order to prove the second part, consider a convergent infinite subsequence x_{k_l} which is not (NCS). Then x_{k_l} converges to a global minimum x^* of problem (P) by Theorem 69 and $J(x^*) = J^*$.

Let $\mathbb{P} = \{k_l, l \in \mathbb{N}\} \subset \mathbb{N}$. Every element of the sequence $(x_k)_{k \in \mathbb{N}}$ satisfies

$$J(x_k) \geq J(x_{s(k)}), \text{ where } s(k) = \max\{\ell \in \mathbb{P} \mid \ell \leq k\}. \quad (6.10)$$

Note that $(x_{k_l})_{k \in \mathbb{N}}$ is a subsequence of $(x_{s(k)})_{k \in \mathbb{N}}$. Since $J(x_{s(k)})_{k \in \mathbb{N}}$ is nondecreasing, bounded from above J^* , we obtain $J(x_{s(k)}) \rightarrow J^*$. By (6.10), $J(x_k) \rightarrow J^*$.

For any sequential cluster point $\bar{x} \in \mathbb{R}^n$ of x_k , we must have $J(\bar{x}) = J^*$. □

6.3 Inexact Algorithm

In this section, we propose a modification of the Warm restart outer approximation algorithm 2. At each iteration, we solve the outer approximation problem OP_k in polynomial time for a fixed dimension $n \in \mathbb{N}$, as discussed in the previous section.

We also address the situation mentioned in the previous section, when the optimal value of the Problem (OP_k) is not known.

The outer approximation Algorithm 3 described in this section finds a ε -solution of Problem (P), for an arbitrary $\varepsilon > 0$. Objective function is $J(x) := \|x\|^2$, (i.e. $z = 0$, a simple translation is performed otherwise). A ε -solution is defined as follows.

Definition 71. Let J^* be the optimal value of a general optimization problem

$$\min_{x \in H} J(x) \quad \text{s.t. } x \in C \quad (\text{GP})$$

where $C \subset H$ is close and $J : H \rightarrow (-\infty, +\infty]$ is proper. $x^\varepsilon \in \mathcal{C}$ is a ε -solution of Problem (P), $\varepsilon \geq 0$, iff

$$x^\varepsilon \in \{x \in \mathcal{C} \mid J(x) \leq J^* + \varepsilon\}. \quad (6.11)$$

At every iteration k of Algorithm 3, the outer approximation (OP_k) takes the form

$$\min_{x \in \mathbb{R}^n} \|x\|^2 \quad \text{s.t. } x \in \mathcal{O}_k.$$

We rewrite Problem (OP_k) as the feasibility problem (5.2) by taking $\alpha = \|x_k\|^2$. Problem (5.2) is solved either exactly, by finding a global minimum (1 and 2a) following Section 5.3, or inexactly, by finding an ε -solution, in accordance with the following subroutine (2b).

Subroutine 1 Outer problem solver for chosen $\varepsilon > 0$.

- 1: If there exists a nonzero direction of recession ($\text{rec } \mathcal{O}_k^\alpha \neq \{0\}$), calculate a solution x_{FP} of the outer approximation problem (5.2), as suggested in Remark 21. Set $x_{k+1} = x_{FP}$.
 - 2: Else if $\text{rec } \mathcal{O}_k^\alpha = \{0\}$, enumerate all the vertices of $\mathcal{O}_k^\alpha$ to solve the feasibility problem (5.2), as suggested in Remark 22. Two situation may appear.
 - (a) If the feasibility problem (5.2) has a solution x_{FP} , set $x_{k+1} = x_{FP}$.
 - (b) Otherwise, the feasibility problem (5.2) has no solutions. Take any point $x_{k+1} \in \{x \in \mathbb{R}^n \mid J(x) = J(x_k) + \varepsilon\}$.
 - 3: **Return** x_{k+1} .
-

Remark 25. From a geometrical point of view, when the feasibility problem (5.2) has no solution, that the polytope $\mathcal{O}_k^\alpha$ is inside $B(0, \sqrt{\alpha})$. Consider the constrained set $\mathcal{O}_k$ which is the complement of the union of interiors of open balls $q_i(x) > 0, i = 1, \dots, m$ (see Section 3.4). when the feasibility problem (5.2) has no solution, the open balls $q_i(x) > 0, i = 1, \dots, m$ completely covers the level set $\text{lev}_{\leq \alpha} J = \{x \in H \mid \|x\|^2 \leq \alpha\}$. Hence, $\mathcal{A}$ (the feasible set of (P)) lies outside of $\text{lev}_{\leq \alpha} J$.

Let m_k be the cost of performing the Subroutine 1. The cost m_k is fixed before we run Subroutine 1 (for example m_k can be the number of constraints of (OP_k)). The definition of $\bar{m}$ is the same of Inexact cutting spheres Algorithm 2.

Algorithm 3 Inexact cutting spheres

Ensure: $x_0 = 0$, $r_0 = 0$, $k = 0$, $\mathcal{O}_{-1} = \mathbb{R}^n$, $m_0 = 0$, $\delta > 0$, $\varepsilon > 0$.

```

1: while  $s_{k-1} = 0$  do
2:    $s_k = 1$ 
3:   if  $I(x_k) \neq \emptyset$  then
4:     if  $m_k \geq \bar{m}$  then
5:       if  $J(x_k) > J(x_{r_k}) + \delta$  then
6:          $r_{k+1} = k + 1$ 
7:         Build  $\mathcal{O}_k$  as in (6.1)
8:          $s_k = 0$ 
9:       end if
10:    else
11:       $r_{k+1} = r_k$  and build  $\mathcal{O}_k$  as in (3.13)
12:       $s_k = 0$ 
13:    end if
14:    if  $s_k = 0$  then
15:      Apply Subroutine 1 with  $\varepsilon$  to find  $x_{k+1}$ 
16:      Return  $x_{k+1}$ .
17:      if  $x_{k+1}$  was calculated at point 2(b) of Subroutine 1 then
18:        Set  $r_{k+1} = k + 1$ .
19:        Set  $\mathcal{O}_k = \{x \in \mathbb{R}^n \mid \|x\|^2 \geq J(x_{k+1})\}$ 
20:      end if
21:    end if
22:     $k \leftarrow k + 1$ .
23:  end if
24: end while

```

The following assumption is needed in order for Inexact cutting spheres Algorithm 3 to be well defined.

Assumption 72. The constant $\varepsilon > 0$ is such that for any $\varepsilon_1 \in (0, \varepsilon)$ $\mathcal{A} \cap \text{lev}_{=J^*+\varepsilon_1} J \neq \emptyset$, where $\text{lev}_{=J^*+\varepsilon_1} J := \{x \in \mathbb{R}^n \mid J(x) = J^* + \varepsilon_1\}$.

The following two propositions illustrates the properties of Inexact cutting spheres Algorithm 3.

Proposition 73. Consider the Inexact algorithm 3 under the Assumption 72. Assume that Algorithm 3 terminates in a point $x_{\bar{k}}$ such that $I(x_{\bar{k}}) = \emptyset$. Two situations can happen.

- i. Algorithm 3 terminates in a finite number of steps and the last iteration $x_{\bar{k}}$ is a global minimum of Problem (P).
- ii. Algorithm 3 terminates in a finite number of steps and the last iteration $x_{\bar{k}}$ is a ε -solution of Problem (P).

Proof. Algorithm 3 returns a point $x_{\bar{k}} \in H$, at an iteration $\bar{k} \in \mathbb{N}$, with $m_{\bar{k}} \leq \bar{m}$. By assumption, it must be $I(x_{\bar{k}}) = \emptyset$, i.e. $x_{\bar{k}}$ is feasible for Problem P. Since $x_{\bar{k}} \in \mathcal{A}$, we have that $J(x_{\bar{k}}) \geq J^*$. Two cases may arise:

- 1) $J(x_{\bar{k}}) = J^*$, then $x_{\bar{k}}$ is a global solution of Problem (P).
- 2) Otherwise, $J(x_{\bar{k}}) > J^*$. In this case the previous restart i.e. $r_{\bar{k}}$ was done by subroutine 2(b). We have $J(x_k) = J(x_{r_{\bar{k}}}) + \varepsilon$ and by Assumption 72 the intersection $\mathcal{A} \cap \text{lev}_{=J(x_{r_{\bar{k}}})+\varepsilon}$ is nonempty. In est., by construction $J(x_{\bar{k}}) >$

$J^* > J(x_{r_{\bar{k}}-1})$. Since, by construction $J(x_{\bar{k}}) = J(x_{r_{\bar{k}}-1}) + \varepsilon$ we obtain that $J^* > J(x_{\bar{k}}) - \varepsilon$, i.e. $J^* + \varepsilon > J(x_{\bar{k}})$. Then, $x_{\bar{k}}$ is a ε -solution of problem (P), which proves the point (ii).

□

Similarly to Algorithm 2, Algorithm 3 generates a sequence $(x_k)_{k \in \mathbb{L}}$ where we denote with $\mathbb{L} \subset \mathbb{N}$ the set of running iterations. Let $\mathbb{K}$ be the set of restart iterations, defined as

$$\mathbb{K} = \{k \in \mathbb{L} \setminus \{0\} \mid r_k \neq r_{k-1}\}. \quad (6.12)$$

So $\mathbb{K}$ is the set of iteration numbers for which lines 7 or 19 in Algorithm 3 are performed. Notice that we have two different types of restart iterations:

- i. The restart iterations performed at line 7 of Algorithm 3, which we denote exact restarts.
- ii. The restart iterations performed at line 19 of Algorithm 3, which we denote inexact restarts.

Proposition 74. Under the assumptions of Theorem 30 and Assumption 72, consider Algorithm 3. Let $(\mathcal{O}_{k-1}, x_k)_{k \in \mathbb{L}}$ be generated by Algorithm 3.

- i. For every $k \in \mathbb{L}$, we have $\mathcal{O}_{k-1} \subseteq \mathcal{O}_{k-2} \subseteq \dots \subseteq \mathcal{O}_{r_k-1}$.
- ii. For every $k \in \mathbb{L}$, $\{\mathcal{A} \setminus \text{lev}_{\leq J^* + \varepsilon} J\} \subseteq \mathcal{O}_{k-1}$.
- iii. $(J(x_k))_{k \in \mathbb{L}}$ is a nondecreasing sequence bounded from above by $J^* + \varepsilon$ and $(J(x_k))_{k \in \mathbb{K}}$ is an increasing sequence.

- iv. For every $k \in \mathbb{L}$, $I(x_k) = \emptyset$ if and only if $x_k \in \mathcal{A}$ and $J(x_k) \leq J^* + \varepsilon$.
- v. Since $\delta > 0$, then
 - (a) for every $k \in \mathbb{L}$, the number of constraints in $\mathcal{O}_{k-1}$ never exceeds $\overline{m}$;
 - (b) there exists $k^* \in \mathbb{N}$ such that $s_{k^*} = 1$ (so, the algorithm “stops in a finite number of steps”);
 - (c) either $x_{k^*} \in \text{lev}_{\leq J^* + \varepsilon} J$ and $x_{k^*} \in \mathcal{A}$, or the resulting lower approximation to $J^* + \varepsilon$ is $J(x_{k^*})$.

Proof.

- i. These inclusions follow from the fact that, between two consecutive restarts, Algorithm 2 accumulates constraints.
- ii. Notice that

$$\mathcal{A} \subseteq \{x \in \mathbb{R}^n \mid (p = r_k, \dots, k), (\forall i \in I(x_p)) L_i(\varphi_i^p, x_p)(x) \leq 0\}, \quad (6.13)$$

and

$$\mathcal{O}_{k-1} := \{x \in \mathbb{R}^n \mid (p = r_k, \dots, k), (\forall i \in I(x_p)) L_i(\varphi_i^p, x_p)(x) \leq 0\} \cap \{x \in \mathbb{R}^n \mid J(x) \leq J_{r_k}\}.$$

By Proposition (73), $J(x_k) \leq J^* + \varepsilon$, $\forall k \in \mathbb{L}$. Hence $J_{r_k} = J(x_{r_k}) \leq J^* + \varepsilon$ and the proof follows.

- iii. In a cumulative step, the non-decreasing property follows from (i). In turn, the cost increases in a restart iteration is due to the last inequality constraint in

(6.1) (exact restart iteration) or Subroutine 1.2b (inexact restart iteration). By Proposition 73, for every $k \in \mathbb{L}$, $J(x_k) \leq J^* + \varepsilon$. Let $(k_1, k_2) \in \mathbb{K}^2$ be such that $k_1 = \max\{k \in \mathbb{K} \mid k < k_2\}$. Then $J(x_{k_2}) \geq J(x_{k_2-1}) > J(x_{r_{k_2}-1}) = J(x_{k_1})$. This shows that $(J(x_k))_{k \in \mathbb{K}}$ is an increasing sequence.

iv. See Proposition 73.

v. (a) This follows from the construction of the algorithm.

(b) $s_k = 1$ is a stopping criterion as we repeat the while loop as long as $s_{k-1} = 0$.

Two situation may appear when we run Algorithm 2.

1. We find a global ε -solution x^* for problem (P), at a iteration $k^* \in \mathbb{N}$.
2. We do not find a global minimizer x^* for problem (P) in a finite number of iterations.

In case 1., there is nothing to prove. Let us show that, in case 2., the algorithm also terminates in a finite number of steps.

Let us consider the iterations when line 7 or line 7 of Algorithm 2 are performed. As shown in Proposition 66.v, since $\delta > 0$, the number of exact restart iterations is finite, where the exact restarts are described at line 7 of Algorithm (3). Since $\varepsilon > 0$, we can repeat the same proof for the inexact restart iterations performed at line 7 of Algorithm (3). Therefore, the inexact restart iterations can be performed only a finite number of times. Hence the number of restarts, exact or inexact is finite, i.e. the cardinality of set $\mathbb{K}$ is finite. The proof proceed as the proof of Proposition

66.v.

(c) The proof follows from (iii) and (73).

□

Chapter 7

Numerical application: the packing problem

7.1 Chapter Outline

The Circular Packing Problem (CPP), introduced in the next Section 7.2, is a NP-Hard problem and it is often reformulated as a QCQP problem. Some references on CPP are (see e.g. Castillo et al. (2008); Addis et al. (2008); Stetsyuk et al. (2016)) and the websites '<http://www.packomania.com/ccineq/>' and '<http://hydra.nat.uni-magdeburg.de/packing/ccis/ccis.html>' are useful resources.

Our aim is to test the Inexact algorithm 3 on a simple instance of CPP and not to compete with already known *ad hoc* methods for finding a solutions of many instances of CPP.

- In Section 7.2, we define CPP and reformulate it in the form of (P).
- In Section 7.3, we illustrates our numerical results. More information on how we solve the outer approximation problem (OP_k) at each iteration can be found in Subsection 7.3.1.

7.2 Reformulating the packing problem in the form of (P)

In this section we introduce the circular packing problem (CPP), which consists in finding the minimal radius r of a circle in the plane, centered in zero, that contains m circles of given radius r_i $i = 1, \dots, m$. The centers (x_i, y_i) of the circles to be packed and the radius r of the packing circle are the variables of (CPP).

The circular packing problem can be modelled as follows:

$$\begin{aligned}
 & \text{Min}_{r,x,y} \quad r^2 \\
 & \text{s.t.} \quad f_i(x, y, r) := x_i^2 + y_i^2 - (r - r_i)^2 \leq 0, \quad i = 1, \dots, m \\
 & \quad \quad f_{ij}(x, y) := -(x_i - x_j)^2 - (y_i - y_j)^2 + (r_i + r_j)^2 \leq 0, \quad 1 \leq i < j \leq m \\
 & \quad \quad r \geq 0.
 \end{aligned} \tag{CPP}$$

The problem (CPP) is a special case of problem (P0). Hence, (CPP) can be cast in the form of (P) as discussed in Section 3.2.

The calculations presented in Section 3.2 can be performed as follows.

The first group of constraints, i.e. $f_i(x, y, r) \leq 0$, $i = 1, \dots, m$, can be rewritten as

$$r \geq \max_{i=1, \dots, m} \{ \|(x_i, y_i)\| + r_i \}$$

where, for $i \in \{1, \dots, m\}$, $\|(x_i, y_i)\|$ is the Euclidean norm for $(x_i, y_i) \in \mathbb{R}^2$.

We can rewrite the packing problem as

$$\begin{aligned} \text{Min}_{x,y} \quad & \max_{i=1,\dots,m} \{(\|(x_i, y_i)\| + r_i)^2\}, \\ \text{s.t.} \quad & f_{ij}(x, y) := -(x_i - x_j)^2 - (y_i - y_j)^2 + (r_i + r_j)^2 \leq 0, \quad 1 \leq i < j \leq m. \end{aligned} \quad (7.1)$$

Let S be the feasible set of (7.1) and $F(x, y) = \max_{i=1,\dots,m} \{(\|(x_i, y_i)\| + r_i)^2\}$. As described in Section 3.2, we need to choose $(\rho, \eta) \in]0, +\infty[\times \mathbb{R}$, such that

$$D = \left\{ (\hat{x}, \hat{y}) \in \text{Argmin}_{(x,y) \in S} F(x, y) \mid F(\hat{x}, \hat{y}) + \eta \geq \frac{\rho}{2} \|(\hat{x}, \hat{y})\|^2 \right\} \neq \emptyset. \quad (7.2)$$

Take $\rho/2 = 1/m$ and $\eta = 0$. For all $(x, y) \in \mathbb{R}^{2m}$,

$$\max_{i=1,\dots,m} \{(\|(x_i, y_i)\| + r_i)^2\} \geq \max_{i=1,\dots,m} \{\|(x_i, y_i)\|^2\} \geq \frac{1}{m} \sum_{i=1}^m \|(x_i, y_i)\|^2 = \frac{1}{m} \|(x, y)\|^2,$$

where $\|(x, y)\|$ is the Euclidean norm of $(x, y) \in \mathbb{R}^{2m}$. By Theorem 20, by introducing a variable $p \in \mathbb{R}$, we can rewrite (7.1) equivalently as

$$\begin{aligned} \text{Min}_{(x,y,p) \in \mathbb{R}^{2m+1}} \quad & \|(x, y, p)\|^2 \\ \text{s.t.} \quad & f_{ij}(x, y) := -(x_i - x_j)^2 - (y_i - y_j)^2 + (r_i + r_j)^2 \leq 0, \quad 1 \leq i < j \leq m, \\ & g(x, y, p) := \max_{i=1,\dots,m} (\|x_i, y_i\| + r_i)^2 - \frac{\|(x, y, p)\|^2}{m} \leq 0. \end{aligned} \quad (\text{CPP.2})$$

After we find a solution (x^*, y^*, p^*) of (CPP.2), we get the optimal disposition of the centers (x^*, y^*) of the circles to be packed in the plane. The optimal radius r^* , which is the minimal radius of the circle centered in the origin that contains the circles of radius r_i $i = 1, \dots, m$, satisfies $r^* = \max_{i=1,\dots,m} \{\|(x_i^*, y_i^*)\| + r_i\}$.

7.3 Numerical experiment with four circles of radius 1

Our experiment consist in a particular instance of the packing problem (CPP.2), where we should pack four circles of radius one. Hence, the dimension of the space is $\mathbb{R}^9$ and the number of weakly convex (quadratic) constraints is $m = 7$.

Since this is a simple instances, it is not difficult to find an optimal position of the centers of the circle in the plane, which can be

$$(x^*, y^*) = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (1, 1, 1, -1, -1, 1, -1, 1, -1). \quad (7.3)$$

Moreover, we calculate, with a gridsearch, that the variable $p \in \mathbb{R}$, which is essential in the formulation of the circular packing problem (CPP.2), takes the optimal value $p^* = 3.914$ for (x^*, y^*) as in (7.3). Therefore, the optimal level set for (CPP.2) is 23.3194.

We start from a random point on $\text{lev}_{=21} J$ and we set $\varepsilon = 3$.

Then, we expect the Inexact algorithm 3 to

- i. establish that no optimal feasible solution for the (CPP.2) exists at $\text{lev}_{=21} J$,
- ii. jump to $\text{lev}_{=24} J$ with a restart iteration,
- iii. find a feasible solution (x^*, y^*, p^*) for (CPP.2) at the current level set, i.e.

$$J(x^*, y^*, p^*) = \|(x^*, y^*, p^*)\|^2 = 24.$$

We ran our experiment on a Microsoft Windows 10 Pro machine with a 3.5 Ghz processor AMD Ryzen 9 3950X 16-Core.

After 4 hours and 12 : 20 minutes, at iteration 1143, Inexact algorithm 3 establishes that no feasible solution for the (CPP.2) exists at $\text{lev}_{=21} J$. The number of constraints at this point is 5231. So a restart iteration happens and x_{1143} is a random point on the level set $\text{lev}_{=24} J$. Then, after 6 minutes, for a total running time of 4 : 18 : 01, at iteration number 2267, the feasible ε -optimal solution

$$(x^*, y^*, p^*) = (1.33713909, 0.58961196, -1.31825987, -0.5798902, \\ -0.55018788, 1.32148933, 0.59123494, -1.32301583, 3.95491652)$$

is found. It is worth to mention that for level sets which are far from optimal level set, the inexact algorithm 3 is much faster to recognizes that no feasible solution for (P) belongs to the level set and perform a restart iteration.

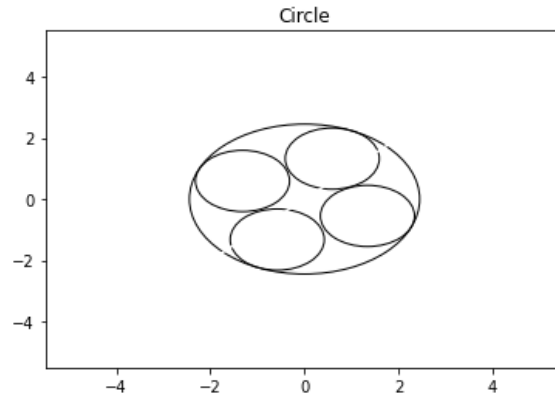


Figure 7.1: (x^*, y^*, p^*) for the (CPP.2) with four circles of radius one.

Computational details for solving the outer approximation problem (OP_k) at each iterations are given in the next subsection.

7.3.1 Computing Subroutine 1

The computational realization of Subroutine 1, which is performed at each step of the inexact algorithm (3) in order to find $(x, y, p)_{k+1}$, is as follows:

- i. We find a point $(x_1, y_1, p_1) \in \mathcal{O}_k^\alpha$, where $\mathcal{O}_k^\alpha$ is the polyhedron defined in (5.2) and $\alpha = J((x, y, p)_k) = \|x_k, y_k, p_k\|^2$, such that $J((x, y, p)_1) = \|x_1, y_1, p_1\|^2 \leq \alpha$.
- ii. The solver [Gurobi Optimization, LLC \(2023\)](#) is called to solve the concave minimization problem

$$\min_{p,x,y} J(x, y, p) = \|p, x, y\|^2 \quad s.t. \quad x \in \mathcal{O}_k^\alpha \quad (7.4)$$

with an early stopping command, such that [Gurobi Optimization, LLC \(2023\)](#) terminates as soon as it finds a point (x_2, y_2, p_2) , feasible for $\mathcal{O}_k^\alpha$ and with $J((x, y, p)_2) = \|x_2, y_2, p_2\|^2 \geq \alpha$.

- iii. If the point (x_2, y_2, p_2) exists, as already discussed in Subsection 5.3, we can find a global solution $(x_{k+1}, y_{k+1}, p_{k+1})$ for the outer approximation problem (OP_k) by scaling (x_2, y_2, p_2) on the line passing for (x_2, y_2, p_2) and (x_1, y_1, p_1) such that $J(x_{k+1}, y_{k+1}, p_{k+1}) = \|(x_{k+1}, y_{k+1}, p_{k+1})\|^2 = \alpha$ (This point of the enumeration corresponds to Subroutine 1, points 1 and 2a).
- iv. Otherwise, [Gurobi Optimization, LLC \(2023\)](#) establishes that the optimal solution $(\bar{x}, \bar{y}, \bar{p})$ of (7.4) has objective value $J(\bar{x}, \bar{y}, \bar{p}) = \|\bar{x}, \bar{y}, \bar{p}\|^2 < \alpha$ and we change level set by setting $(x_{k+1}, y_{k+1}, p_{k+1})$ equal to a random point on the

level set $J_k + \varepsilon = J((x, y, p)_k) + \varepsilon$ (This point of the enumeration corresponds to Subroutine [1](#), point $2b$).

Conclusions

- The subdifferentials for weakly convex functions play a central role in this thesis. In Chapter 2, following [Bednarczuk et al. \(2023b\)](#), we study sum rules for the proximal subdifferential (9) and ε -proximal maps for weakly convex functions, which are useful tools for proving the convergence of splitting methods for unconstrained weakly convex problems, see e.g. [Bednarczuk et al. \(2023c\)](#).

On the other hand, the Φ -subdifferential, Definition 8, is the main tool for building outer approximation sets for the feasible set $\mathcal{A}$ of (P), see Section 2.4.

- The main result of this work is Theorem 30, where we prove that the sequence $(x_k)_{k \in \mathbb{N}}$ produced by Cutting sphere algorithm 1 possesses a convergent subsequence and every cluster point of $(x_k)_{k \in \mathbb{N}}$ is a global minimum of (P)

The Cutting sphere algorithm 1 is the first algorithm which finds a global minimum of (P).

- The Cutting sphere algorithm 1 solves (OP_k) at each iteration, which is a [QCQP](#) problem.

With the aim of establish which is the best way to find a global minimum of (OP_k) , we study the [SDP](#) relaxation for [QCQP](#) in Section 4.3.2, Lagrangian

duality for QCQP in Section 4.4, hidden convexity in Section 4.5 and the S-Lemma in Section 4.6. We study the relations between the notions listed above and sufficient and necessary conditions for global optimality of QCQP. Our original contribution on these topics are from the paper [Bednarczuk and Brucola \(2023\)](#), which studies the S-Lemma and the KKT conditions for QCQP and the subclass S-QCQP, which is relevant for (OP_k) .

- In Chapter 6, we produce two algorithms (the cutting sphere algorithm 2 and the inexact algorithm 3) which serve as computational realizations of the Cutting sphere method.
- We test the inexact algorithm 3 on Chapter 7 on a simple instance of CPP.
- We have the goal to improve the computational aspects of the new Cutting sphere method in future researches.

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