Affine continuous cellular automata and their role in solving density classification problems

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Abstract

Cellular Automata (CAs) are dynamical systems, where time and space are discrete. Most often, cell states take values from a finite set. State changes in subsequent time steps are defined using a local rule that considers the states of the nearest neighbors. Despite such a simple definition, CAs are characterized by interesting and complex properties and are therefore used to model different phenomena, especially physical ones. Theoretical research most often concerns the simplest two-state CAs, which are defined on a one- or two-dimensional finite grid with periodic boundary conditions.

This dissertation presents the results of my research on Affine Continuous Cellular Automata (ACCAs), in which cell states take values in the range $[0, 1]$, and the local rule is an affine function in each variable. Although CAs defined in this way are the simplest possible extension of binary CAs to continuous automata, it turns out their dynamics are much more complex and interesting. To justify this statement, the paper presents the results of detailed studies of outer totalistic ACCAs, showing that within this very trivial subset of ACCAs all sensitivities found in the dynamics of CAs can be observed.

The primary focus of my work is on investigating how ACCAs can be used to solve the famous theoretical problem in cellular automata known as the density classification problem (DCP) or majority voting problem. In the classical formulation, DCP looks for a binary CA that can correctly classify each binary initial configuration: if it contains more 1s than 0s, then as the final configuration we get a homogeneous configuration consisting of only 1s, and if it contains more 0s than 1s, then the final configuration we get a homogeneous configuration consisting of all zeros. Unfortunately, it has been proven that there is no binary CA that can correctly classify any initial configuration of any length, *i.e.*, the classic version of DCP has no solution. Therefore, a natural question arises whether relaxing the requirements for DCP, *e.g.* allowing non-binary CAs as solutions or changing the initial conditions, will allow us to find a classifier that works correctly on all binary initial configurations.

In my scientific work, I deal with the above issue in the context of ACCAs. The automata I study are one- and two-dimensional with a neighborhood radius of 1 and periodic boundary conditions (in the case of two-dimensional ACCA, the von Neumann neighborhood is used). The presented comprehensive research, descriptions, experiments, as well as analytical proofs show that while for ACCAs there is no single local rule allowing for the correct classification of the density of all binary initial configurations with the classic formulation of the final condition, its slight modification allows finding a numerous subset of rules ACCAs that turn out to be correct solutions. It appears that these ACCAs are responsible for conserving density.

Furthermore, I am looking into the potential of using ACCAs to address classical DCP on a fixed-size grid in one or two dimensions. It seems that there are solutions available in this situation.

Most of the chapters in my doctoral dissertation were published in peer-reviewed articles [1–5] and presented at international scientific conferences. This dissertation not only collects all these articles and presentations but also presents the results of additional simulations and calculations. Unlike the chronology of publications, the dissertation presents the results in an order explaining first the basics of the subject, moving on to the details of the issues. All published articles and presentations result from teamwork. I am the author of all visualizations, simulations, experiments in published articles and presentations. I analyzed the results collected from experiments, which were the basis for analytical proofs and claims. Proofs of the theorems were achieved thanks to cooperation with Barbara Wolnik [1–5] and Antoni Augustynowicz [5]. The remaining co-authors of the articles supported the research through discussion, verification of the correctness of the obtained results, and verification of the articles.

Content of individual chapters

Below, I summarize the individual chapters of the work.

Chapter 1 - Introduction

The initial section of the doctoral thesis covers a brief history and uses of CA and gives an overview of the rules for the Game of Life, which is a highly popular two-dimensional cellular automaton. I wrote this chapter especially for the needs of the following dissertation and it is only an introduction to the general topic of CAs.

Chapter 2 - Preliminaries

Chapter two contains the definitions necessary for discussing the research and its outcomes. These are definitions generally known in the literature and used in my publications, however, for the purposes of this dissertation, the notation was unified. The chapter begins by presenting and defining a one-dimensional grid ($\mathcal{C}_1$) and a two-dimensional rectangular grid ($\mathcal{C}_2$) with periodical boundary conditions:

$$\begin{aligned} \mathcal{C}_1 &= (\mathbb{Z}/n\mathbb{Z}) = \{0, 1, \ldots, n-1\} , \\ \mathcal{C}_2 &= (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z}) = \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, m-1\} , \end{aligned}$$

where n and m are natural numbers greather than 2.

We define a set of states $\mathcal{Q}$ where the main focus is on binary state set $\mathcal{Q}_B = \{0, 1\}$ and real-value state set $\mathcal{Q}_{\mathbb{R}} = [0, 1]$. For the above sets of states we define set of all configurations $\mathcal{X}$ for a given $\mathcal{C}$ as a set of all transformations from $\mathcal{C}$ into $\mathcal{Q}$. For a configuration $\mathbf{x} \in \mathcal{X}$ as x_i we define the state of i-th cell (similar for the two-dimensional case $x_{i,j}$). A key parameter in my research is configuration density $\rho(\mathbf{x})$ being the average of all cell states.

Since only binary initial configurations are important in the research, I describe a way of reducing the set of such configurations to "significantly different", *i.e.*, those that are not shifts of each other. While rarely discussed in the literature, this fact greatly influences the calculations in subsequent chapters.

My considerations focus on the neighborhood with a radius $r = 1$, *i.e.*, where the cell has only insight into the states of its nearest neighbors. For two-dimensional CAs this means the von Neumann neighborhood. All the cells are updated at each discrete time step $t \in \{1, 2, \ldots\}$. The state is changed synchronously: each cell uses the same local rule $\mathbb{f}$ in the neighborhood. As a result we get a global rule $\mathbb{F}$ for the configuration. The local rule $\mathbb{f}$ we can represent in a tabular form LUT (LookUp Table) where the coefficients l_i define a transition function for a given neighborhood. This representation simplifies calculations and implementation of algorithms.

In the next part of this chapter I present the most common CAs in the literature - Elementary CAs (ECAs), a one-dimensional, binary CAs with the radius one.

At the end, I introduce the definition of continuous CAs (CCAs) and I quote well known local rules $\mathbb{f}_d$, being the average value of the state of the cell and its neighbors. I also present a method of dynamics visualization of such automata.

Chapter 3 - Affine Continuous Cellular Automata

This chapter focuses on affine CCAs (ACCAs), which are involved in research and are a generalization of ECAs. First, the fact saying that when a function is affine in some variable, it must also be monotonic in that variable. This obvious fact is crucial because it guarantees that the ACCA under consideration will not go beyond the set of states specified at the beginning.

In the first part of this chapter, I present the definition of one-dimensional ACCAs, giving the local rule $\mathbb{f}$ by the coefficients l_i from the LUT table:

$$\begin{aligned} \mathbb{f}(x, y, z) = & l_0(1-x)(1-y)(1-z) + l_1(1-x)(1-y)z \\ & + l_2(1-x)y(1-z) + l_3(1-x)yz + l_4x(1-y)(1-z) \\ & + l_5x(1-y)z + l_6xy(1-z) + l_7xyz . \end{aligned}$$

Next, I define specific subsets of ACCAs. The first type is known as outer-totalistic CAs, where the rule is based on the state of a cell and the total sum of its neighbors' states. Chapter 5 contains the research on these rules.

Next, I give the definitions of density-conserving automata. It is a family of CCAs, where for each configuration $\mathbf{x} \in [0, 1]^\star$ meet the condition

$$\rho(\mathbf{x}) = \rho(\mathbb{F}(\mathbf{x})) .$$

For one-dimensional ACCAs, the LUT for such rules can be reduced to three parameters.

$$(l_0, l_1, l_2, l_3, l_4, l_5, l_6, l_7) = (0, a, c, 1-b, 1-a-c, 1-c, b+c, 1) , \tag{1}$$

where $a, b, c \in [0, 1]$ and $a, b \leq 1 - c$. This reduction allows for more detailed research as well as a visual representation of the arrangement of individual rules. The study of these types of rules is the main focus of Chapter 6.

In the second part of this chapter, I extend the definitions to two-dimensional ACCAs. I start by defining a local rule $\mathbb{f}$ for two dimensions

$$\begin{aligned}\mathbb{f}\left(\begin{smallmatrix} & x & \\ y & z & u \\ & v & \end{smallmatrix}\right) =& l_0(1-x)(1-y)(1-z)(1-u)(1-v)\\ &+ l_1(1-x)(1-y)(1-z)(1-u)v\\ &+ l_2(1-x)(1-y)(1-z)u(1-v)\\ &\vdots\\ &+ l_{30}xyzu(1-v)\\ &+ l_{31}xyzuv\ .\end{aligned}$$

After that, I identify various kinds of two-dimensional ACCAs. Specifically, I introduce density-conserving two-dimensional ACCAs and examine the collection of these rules (which currently require eight separate parameters for characterization). I also highlight symmetries present in this set, which enable us to unite specific rules. Additionally, I introduce the definition of density-preserving ACCAs whose local rule does not depend on y and v:

$$\mathbb{f}\left(\begin{smallmatrix} & x & \\ y & z & u \\ & v & \end{smallmatrix}\right) = (1-b-d)x + dz + bu\ ,$$

where $b, d \in [0,1]$ and $b+d \leq 1$.

Later in this chapter, I define rules that are symmetrical with respect to a 90-degree rotation, *i.e.*, those whose local rule satisfies:

$$\mathbb{f}\left(\begin{smallmatrix} & x & \\ y & z & u \\ & v & \end{smallmatrix}\right) = \mathbb{f}\left(\begin{smallmatrix} & y & \\ v & z & x \\ & u & \end{smallmatrix}\right)\ ,$$

and I present the LUT conditions for such rules. Detailed research on this type of rules is outlined in Chapter 8.

Chapter 4 - The Density Classification Problem

In this chapter I explain definitions related to the density classification problem. First, I give a definition of DCP, saying that a global CCA rule $\mathbb{F}$ solves the classical DCP for a given treshold $\rho_0 \in (0,1)$, when for any initial configuration $\boldsymbol{x} \in \{0,1\}^\star$ the following criteria are met:

$$\begin{aligned}\rho(\boldsymbol{x}) < \rho_0 \quad &\Rightarrow \quad \lim_{t\to\infty} \mathbb{F}^t(\boldsymbol{x}) = (0,0,\ldots,0)\ ,\\ \rho(\boldsymbol{x}) > \rho_0 \quad &\Rightarrow \quad \lim_{t\to\infty} \mathbb{F}^t(\boldsymbol{x}) = (1,1,\ldots,1)\ .\end{aligned}$$

I also clarify that for ACCAs there is no rule that solves DCP for any initial configurations of arbitrary length. In Chapter 7, experiments and results for this problem are detailed for one-dimensional ACCAs, while Chapter 9 covers the two-dimensional aspect.

In the second part, I provide the definition of *relaxed* DCP for ACCAs. It is saying that a global CCA rule $\mathbb{F}$ solves the relaxed DCP for a given threshold $\rho_0 \in (0,1)$, if for each initial configuration $\boldsymbol{x} \in \{0,1\}^\star$ the following conditions are met:

$$(\exists T)\left(\forall \boldsymbol{x} \in \{0,1\}^{\mathcal{C}}\right)(\forall t \geq T)\left(\begin{cases}\rho(\boldsymbol{x}) < \rho_0 \Rightarrow \mathbb{F}^t(\boldsymbol{x}) \in [0,\rho_0)^{\mathcal{C}}\\ \rho(\boldsymbol{x}) > \rho_0 \Rightarrow \mathbb{F}^t(\boldsymbol{x}) \in (\rho_0,1]^{\mathcal{C}}\end{cases}\right)\ .$$

Experiments and findings are detailed in Chapter 6 for a single ACCA and in Chapter 8 for two-dimensional ACCAs.

Chapter 5 - Dynamics of One-Dimensional Outer-Totalistic ACCAs

The results of this chapter were published in [5]. They concern a comprehensive analysis of one-dimensional, outer totalistic ACCAs that are legal, *i.e.*, those whose LUT coefficient l_0 is set to 0. Each such local rule is given by three parameters as follows:

$$\mathbb{f}(x,y,z) = (d-c-b)(x+z)y + b(x+z) + cy\ , \tag{2}$$

where $0 \leq b \leq \frac{1}{2}$, $0 \leq c \leq 1$ i $c \leq 2d \leq c+1$. Thanks to this, the set of all such rules can be easily parameterized with parallelepiped points and presented graphically. The experiments allowed us to test each of such rules and revealed the existence of various types of "sensitivity" to changes in the initial conditions:

- sensitivity to slight changes in the parameters of a local rule: a small change in the values of the parameters b, c, d can significantly change the properties of the rule,

- sensitivity to the change of a single value in an initial configuration: changing even one value in the initial configuration can have a huge impact on the dynamics,
- sensitivity to the change of the number of cells in the grid: changing the length of the grid even by one cell can have a significant impact on the dynamics.

After conducting the experiments, the results were classified into groups with similar behaviors. The following subsections focus on individual groups, indicating their location in the mentioned parallelepiped and supporting the validity of the calculations with analytical proofs.

Chapter 6 - Solving the Relaxed One-Dimensional DCP

The results of this chapter were published in [1, 2]. Simulations, supported by analytical proofs, are shown here for one-dimensional ACCAs solving relaxed DCP. The local rules of these ACCAs have the following general form:

$$\mathbb{f}(x, y, z) = (a + b + c - 1)(x - z)y + (1 - a - c)x + cy + az \ , \tag{3}$$

where $a, b, c \in [0, 1]$ and $a, b \leq 1 - c$. Using three parameters allows us to visualize the arrangement of all such rules in three-dimensional Cartesian space. In the first part of the chapter, I present a detailed description of the dynamics of each such rule. The next one focuses on experiments involving finding the best rule that solves the problem, requiring the smallest number of iterations for given configurations and threshold value $\rho_0 \in (0, 1)$. I first focus on $\rho_0 = 0.5$ and 25-cell configurations, while also considering some 149-cell configurations. In the rest of the experiments, I look for a rule with the minimum number of needed configuration iterations to correctly classify the relaxed DCP, taking into account threshold values $\rho_0 \in (0, 1)$ with a step of 0.001 and configurations of length 23 cells.

Rozdział 7 - Solving the Fixed-Length One-Dimensional DCP

The results of this chapter were published in [3]. In this chapter I focus on one-dimensional ACCAs that can solve the classical DCP for fixed-length configurations. The LUT of such ACCAs must mainly meet the following criteria

$$\begin{cases} l_0 = 0 \\ l_1 = a \\ l_2 = c \\ l_3 = 1 - b \\ l_4 = 1 - \eta - a - c \\ l_5 = 1 + \gamma - c \\ l_6 = \epsilon - \gamma + b + c \\ l_7 = 1 \end{cases} \text{, where } \begin{cases} a, b, c \in [0, 1] \\ \gamma \leq c \leq 1 + \gamma \\ -\eta \leq a + c \leq 1 - \eta \\ \gamma - \epsilon \leq b + c \leq 1 + \gamma - \epsilon \end{cases}$$

These rules can be displayed in a three-dimensional Cartesian space for a visual representation of each rule. Sufficient "stopping conditions" for the simulation are shown, *i.e.*, conditions whose allows the simulation to end as we already know what its result will be. This crucial condition enables computer simulations to be more efficient. The simulation part focuses on configurations with a length of 23 cells. Additionally, six cases of unique parameters η, ϵ, γ are presented, containing ACCAs rules solving the classical DCP for the specified configuration.

Chapter 8 - Solving the Relaxed Two-Dimensional DCP

In this chapter I present the results of a search for rules solving relaxed DCP for two-dimensional ACCAs. These results were published in [4]. I present here the experiments and results for the configuration 5×5 and a subset of configurations 23×23. I only consider configurations that have the initial density closest to 0.5. I am looking for rules that can solve a given problem faster than the rule $\mathbb{f}_d$. Additionally, I examine the rules presented in Chapter 3 - symmetrical due to the 90-degree rotation, and independent of y and v.

Chapter 9 - Solving the Fixed-Length Two-Dimensional DCP

This chapter presents results on the search for two-dimensional ACCAs capable of solving classical DCP. These rules are searched by an evolutionary algorithm. First, I present the history and basics of evolutionary algorithms, with an emphasis on the differential evolutionary algorithm that I use here. In the following sections, I focus on finding rules that solve DCP for two-dimensional configurations (3×3), (5×3), (5×5). I point out the problems I encounter and the improvements I use in my experiments. This chapter was the only one not published and only partially presented at the conference.

Chapter 10 - Conclusion

This chapter is a summary of the work and a discussion of directions for future research.

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