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Report

on the thesis submitted in partial fulfillment of the requirements
for the degree of doctor of philosophy by The Hyng Tran

ABSTRACT CONVEXITY: DUALITY, PARTIAL-DUAL AND PROXIMAL BASED ALGORITHMS

The presented dissertation deals with certain optimization problems. It is very extensive. It contains two short introductory chapters. Chapters 3 and 4 constitute Part I: 'Duality for composite abstract convex optimization problems'. Chapters 5 and 6 constitute Part II: 'Algorithms for composite optimization problems involving $\Phi_{lsc}^{\mathbb{R}}$ -convex functions'. The thesis concludes with chapter 7 containing ideas of possible future research and the bibliography containing 97 references.

Part I of the thesis is dedicated to finding necessary and sufficient conditions for zero duality gap for conjugate duality of problem

$$(\text{CP}) : \inf_{x \in X} f(x) + g(Lx)$$

and

$$(\text{DCP}) : \sup_{\psi \in \Psi} - \sup_{x \in X} \{-\psi(Lx) - f(x)\} - g_{\Psi}^*(\psi)$$

in the context of abstract convexity: X is a nonempty set, Y is a vector space, $f : X \rightarrow (-\infty, +\infty]$, $g : Y \rightarrow (-\infty, +\infty]$, $L : X \rightarrow Y$, Φ is a set of elementary functions $\phi : X \rightarrow \mathbb{R}$, Ψ is a set of elementary functions

$\psi : Y \rightarrow \mathbb{R}$, both classes of functions are closed under addition of constants, $0 \in \Phi; 0 \in \Psi$ and $g_{\Psi}^*(\psi) = \sup_{y \in Y} \{\psi(y) - g(y)\}$.

The author proves in Theorem 3 that the sufficient and necessary condition for zero duality gap is:

(i3) For every $\varepsilon > 0$ there exist $x_{\varepsilon} \in X, \psi_{\varepsilon} \in \partial_{\varepsilon, \Psi} g(Lx_{\varepsilon})$ such that $(f + g \circ L)_{\Phi}^*(0) \leq -f(x_{\varepsilon}) + \varepsilon$, where $\partial_{\varepsilon, \Psi} g(Lx_{\varepsilon}) = \{\psi \in \Psi \mid g(y) - g(Lx_{\varepsilon}) \geq \psi(y) - \psi(Lx_{\varepsilon}) - \varepsilon, \forall y \in Y\}$.

In Theorem 4 condition (i3) is replaced by stronger sufficient condition (i4): For every $\varepsilon > 0$ there exist $x_{\varepsilon} \in X, \phi_{\varepsilon} \in \partial_{\varepsilon, \Phi} f(x_{\varepsilon}), \psi_{\varepsilon} \in \partial_{\varepsilon, \Psi} g(Lx_{\varepsilon})$ such that $\forall z \in X : \phi_{\varepsilon}(z) + \psi_{\varepsilon}(Lz) \geq -\varepsilon$ and $\phi_{\varepsilon}(x_{\varepsilon}) + \psi_{\varepsilon}(Lx_{\varepsilon}) \leq \varepsilon$.

Condition (i4) is also necessary if for every $\varepsilon > 0$ there exists $\psi_{\varepsilon} \in \Psi$ such that $(f + \psi_{\varepsilon} \circ L)_{\Phi}^*(0) + g_{\Psi}^*(\psi_{\varepsilon}) - \varepsilon/2 < \inf_{\psi \in \Psi} (f + \psi \circ L)_{\Phi}^*(0) + g_{\Psi}^*(\psi)$ and $-\psi_{\varepsilon} \circ L \in \Phi$.

Theorems 3 and 4 are main results of Part I of the thesis.

Subsequently the author provides the reader with interesting examples of applying his conditions for zero duality gap to specific classes Φ and Ψ :

$$\begin{aligned} \Phi_{conv} &= \{\phi : X \rightarrow \mathbb{R} \mid \phi(x) = \langle u, x \rangle_X + c, u \in X^*, c \in \mathbb{R}\}, \\ \Psi_{conv} &= \{\psi : Y \rightarrow \mathbb{R} \mid \psi(y) = \langle v, y \rangle_Y + d, v \in Y^*, d \in \mathbb{R}\}, \\ \Phi_{lsc}^a &= \{\phi : X \rightarrow \mathbb{R} \mid \phi(x) = -a\|x\|^2 + \langle u, x \rangle_X + c, u \in X^*, c \in \mathbb{R}\} \\ \text{and } \Psi_{lsc}^b &= \{\psi : Y \rightarrow \mathbb{R} \mid \psi(y) = -b\|y\|^2 + \langle v, y \rangle_Y + d, v \in Y^*, d \in \mathbb{R}\}. \end{aligned}$$

In Part II, chapter 5.1 the author introduces new proximal operator $\text{prox}_{\gamma f}^{lsc, \mathbb{R}} : X \rightrightarrows X$, $\text{prox}_{\gamma f}^{lsc, \mathbb{R}}(x) := (J_{\gamma} + \partial_{lsc}^{\mathbb{R}} f)^{-1} J_{\gamma}(x)$ where $J_{\gamma}(x) := \partial_{lsc}^{\mathbb{R}} g_{\gamma}(x)$, $g_{\gamma}(x) = \frac{1}{2\gamma} \|x\|^2$. Next the author proves two main theorems of Part II. Theorem 10 states that *if X is a Hilbert space and $f : X \rightarrow (-\infty, +\infty]$ is a proper $\Phi_{lsc}^{\mathbb{R}}$ -convex function then:*

- (a) *Every global minimizer of $\min_{x \in X} f(x)$ is a fixed point of $\text{prox}_{\gamma f}^{lsc, \mathbb{R}}$.*
- (b) *Every fixed point x_0 (i.e. such that $x_0 \in \text{prox}_{\gamma f}^{lsc, \mathbb{R}}(x_0)$) is critical (i.e. $(\exists a_0 \in \mathbb{R} \forall x \in X) f(x) - f(x_0) \geq -a_0 \|x - x_0\|^2$).*
- (c) *If a fixed point x_0 satisfies one of the following conditions*
 - (i) $(-\frac{1}{2\gamma}, 0) \in \partial_{lsc}^{\mathbb{R}}(f + \frac{1}{2\gamma} \|\cdot\|^2)(x_0)$, *or*
 - (ii) $\exists \phi_1, \phi_2 \in J_{\gamma}(x_0), \phi_1 - \phi_2 = (a_1 - a_2, u_1 - u_2) \in \partial_{lsc}^{\mathbb{R}} f(x_0), a_1 \leq a_2$*then x_0 is a global minimizer of f .*

Theorem 11 characterizes the elements of $\Phi_{lsc}^{\mathbb{R}}$ -proximal map. Theorems 10 and 11 are main theoretical results of Part II of the thesis. Subsequently, the author formulates and investigates the convergence of $\Phi_{lsc}^{\mathbb{R}}$ -proximal point algorithm, $\Phi_{lsc}^{\mathbb{R}}$ -proximal Forward-Backward algorithm and primal-dual algorithm for weakly convex functions.

The author was not able to avoid a number of misprints. There are some examples:

1. In Example 3, p.16 it should be 'Let X be a Hilbert space...'.
2. In a paragraph before Proposition 2, p.13, it should be 'This fact motivates...'.
3. After formula (3.8), p.11, it should be ' $\partial_{\Phi} f$ is a set-valued mapping...'.
4. In Corollary 1, p.47, in the assumption, Hilbert spaces can be replaced by Banach spaces.
5. In Example 10, p.51, it should be $X = \mathbb{R}^2$.
6. In Remark 5, p.45, it should be 'the condition (i) of Theorem 4 and Theorem 5'.
7. In Remark 7, p.60, it should be 'that condition (ii) of Theorem 6'. and at the end 'can be found in Example 8(iii)'.
8. In Corollary 2, pp.60-61, conditions denoted by (i), (ii), (iii) and (iv) should be denoted by (a), (b), (c) and (d).
9. In the last paragraph of Remark 8, pp.64-65, instead of '...convex spaces. For lower...' it should be '...convex spaces, for lower...'.
10. In the paragraph after Lemma 3, p.70, it should be 'observe that in Theorem 8...'.
11. The last sentence of Theorem 9, p.71 should be reformulated.

Numerous figures in the dissertation are too small to read comfortably. However, I consider all these shortcomings minor and I do not request a revised version of the thesis to be prepared.

Mr. The Hung Tran's thesis is interesting but not easy to read. His results are original and valuable. The thesis is written clearly and carefully. Reasoning and calculations are, as far as I can say, correct. Results from chapters 3 and 4 were already published together with Prof. Dr. Hab Ewa Bednarczuk in 'Duality for composite optimization problem within the framework of abstract convexity'. In: Optimization 72.1 (2023), pp. 37–80.

The dissertation, despite certain defects, demonstrates the ability of Mr. The Hung Tran to conduct research work and his general theoretical knowledge in the field of optimization theory.

In my opinion the thesis submitted by Mr. The Hung Tran satisfies legal requirements for awarding him with the degree of philosophy doctor (art. 187 Act of July 20th, 2018 - Law on higher education and science (Dz.U.2018 poz. 1668 z późn. zm.)). Therefore, I am requesting Scientific council of Systems Research Instytut of Polish Academy of Sciences to allow Mr. Tran to proceed to further stages of the procedure for awarding him the doctoral degree.