AKADEMIA GÓRNICZO–HUTNICZA
IM. STANISŁAWA STASZICA W KRAKOWIE

Wydział Elektrotechniki, Automatyki, Informatyki i Inżynierii Biomedycznej

KATEDRA AUTOMATYKI I ROBOTYKI
Laboratorium Analizy i Wspomagania Decyzji
Kierownik: Prof. dr hab. inż. Andrzej M.J. Skulimowski

REVIEW of the DOCTORAL DISSERTATION

submitted by **Mr. The Hung Tran**

in the partial fulfilment of the requirements for the doctoral degree, entitled

„Abstract Convexity: Duality, Primal-Dual Proximal Based Algorithms"

Supervisor: **Prof. dr hab. Ewa Bednarczuk**

Co-Supervisor: **Prof. Dr. Dirk Lorenz**

I. Formal background

This review was prepared in relation to the reviewer appointment decision of the Scientific Council of the Systems Research Institute of the Polish Academy of Sciences in Warsaw (Instytut Badań Systemowych PAN w Warszawie; IBS PAN) of June 28, 2024, based on the contract signed on August 16, 2024 with the IBS PAN represented by Vice Director for Research Matters, dr. hab. inż. Jan W. Owsiński. The reviewed dissertation was assigned by the Scientific Council of the IBS PAN to the scientific discipline "Automatic Control, Electronics, Electrical Engineering and Space Technologies".

The legal basis for proceeding this review are contained in the following legal acts:

- Regulation of the Minister of Science and Higher Education of September 20, 2018, on fields of science, scientific disciplines, and artistic disciplines, Journal of Laws 2018, item 1818, with later amendments.
- Law on Higher Education and Science of July 20, 2018, Article 187, Journal of Laws of 2018, item 1668, with later amendments, unified text published on October 3, 2024. (https://isap.sejm.gov.pl/isap.nsf/download.xsp/WDU20180001668/U/D20181668Lj.pdf)

The organisational aspects of further proceeding the review are regulated in the IBS PAN decision of June 28, 2024 concerning the procedure of granting the doctoral degrees by the Scientific Council of the IBS PAN.

The doctoral candidate, Mr. The Hung Tran is a graduate of Computer Science, thus he is eligible for granting him a doctoral degree according to the legal acts above. The reviewed manuscript was checked by the Unified Antiplagiarism System (JSA), with the overall assessment "low similarity", test No. 2867684, check No. 2956139. The report from the above check was made available to the reviewer on September 12, 2024. According to the IBS PAN doctoral procedure, the JSA report was not approved yet by the Supervisors at that time.

II. The structure, goal, and the assessment of main results of the dissertation

The Author is concerned with the notion of abstract convexity introduced by Rubinov (cf. the book [82] and two papers [81],[83] cited in the dissertation), its properties, further extensions, and applications in global optimisation. The essence of the presented approach is the use of suitably formulated dual problem, then the construction of a conjugated dual problem, followed by further research on formulating the optimality conditions for primal and dual problems. The title of the dissertation well characterises its content.

The structure of the dissertation is presented as follows. The substantial results are contained in four chapters, grouped in two parts: "Duality for Composite Abstract Convex Optimisation Problems" (Chapters 3 and 4) and "Algorithms for Composite Optimisation Problems involving Φ_{lsc}^{R}-convex functions" (Chapters 5 and 6). The latter are certain simple quadratic functions defined on a Hilbert space. The first part is dedicated to issues concerning the theoretical foundations of the optimality conditions for problems formulated in this dissertation. The second part contains results obtained for a further class of elementary functions that define abstract convexity. This part presents also computational results and solutions to some numerical test problems. Both parts contain 172 pages (pp. 9-181).

The two main parts of the dissertation are preceded by a five-page "Introduction", where the Doctoral Candidate presents the scope of work and its workflow. In the final section „Conclusion and Future Works", the Author briefly (3 pages) discusses and justifies the solution methods developed and obtained results. Additionally, the dissertation includes a bibliography (97 items) and the list of figures with 19 items. The above structure of the main parts of dissertation is inherited from the logical sequence of four articles and preprints corresponding one-to-one to each chapter in the main two parts.

Neither the goal of the dissertation nor the research hypothesis to be proven were clearly formulated. From the remarks in the "Introduction" and from the overall contents of the dissertation it follows that it was aimed to apply properties of abstract convexity and related concepts, such as abstract conjugate and abstract subdifferential to formulate optimality conditions. The key idea behind this approach was formulating a suitable dual problem for a composite minimization problem. Then, conjugate dual problems allowed the Author to

investigate optimality conditions for primal and dual problems and get zero duality gap and strong duality.

The main results of Part I reside in Chapter 4: The proofs of the key Theorems 3 and 4 on sufficient and necessary conditions for zero duality gap in Section 4.2 are correct. The relations between conjugate dual and Lagrange dual have been discussed in Section 4.3. In Chapter 5 (Part II), the proximal operator for the Φ_{lsc}^{R}-convex functions is constructed. The convergence of the proximal point algorithm in proved in Theorem 13, which allows to further extend the algorithm and apply it to a large class of functions Chapter 6 introduces a primal-dual algorithm for weakly convex functions and presents a number of numerical examples.

The overall results show that abstract convexity with quadratic elementary functions allows to formulate abstract conjugate dual problems to general composite minimization problem with nonconvex objective functions. Together with the constructions of abstract conjugate and abstract subdifferentials, the Author proved zero duality gap property and formulated constructive algorithms to compute the optima. The mathematics is generally correct, with some opportunities to simplify the notation and improve the clarity of presentation left for further publications. Generally, the results which were already published in joint papers with the Supervisor present a high mathematical culture and can be a basis for further high-quality research.

In conclusion, despite some critical remarks, the intention of which is to indicate the need to continue and improve the results presented in the dissertation, the results obtained by the Doctoral Candidate meet the requirements for doctoral dissertations in the broadly understood field of Automatic Control. The Author has demonstrated that his mathematical and computational skills allow him to solve complex optimisation problems and to publish these results in recognized scientific journals, as evidenced by two published articles indexed in the bibliographic databases Web of Science and Scopus.

III. Comments, hints, and remarks

3.1 General comments

The critical comments below are to be consider together with the overall positive assessment of the results presented in Section II above.

1. While the research objective can be considered as achieved, the scope and mode of applicability of the developed algorithms require some reflection. In particular, an attempt to demonstrate, through comparisons, the situations in which the developed algorithms outperform results achieved by other methods would be beneficial to the research community.
2. The dissertation does not contain a comprehensive presentation of preliminaries or background information on the topics concerned. Specifically, there is no information about non-convexity and non-convex optimisation in general. No links and relations between abstract convexity and generalised convexity are indicated.
3. Computational consequences of proven facts are weak points of this dissertation. Specifically, Algorithm 1 (p.99), Alg. 2 (p.107), and Alg. 3 (p.109) are presented in an

incomplete way, for example: there are no hints how to select the starting points and what is their impact on the efficiency of computations. Furthermore, no stopping criterion was defined, "For" is used as the keyword in open loops instead of "While". Alg. 4 (p.113) does not even define any computation flow and scope. Section 5.5 contains some numerical computations, but the parameters admitted there are not justified, for example number of iterations or update rule for a_n. Similarly laconic are definitions of dual algorithms in Chapter 6, for example in Alg. 5 (p.149) and Alg. 6 (p.158) it is not clear which parts of the expressions defining x_{n+1} and y_{n+1} are minimised. Furthermore, the computation scope, the stopping criterion or the number of iterations are undefined. Alg. 7 (p.163) suggests an infinite number of iterations ($n \in N$). It has not been specified how the numerical examples were calculated and what was the code used for that.

4. The source codes were not made available for review, which makes the reproducibility check difficult. As the only information, Matlab is mentioned as the computing environment. Hence it follows an insufficient guidance as regards selecting the computational strategy most appropriate for a specific optimisation problem.
5. The selection of references is appropriate, some missing works are recommended below. Further minor shortcomings of editing the list of references are pointed out in the section "Specific remarks" below.
6. The lack of a list of symbols used in the dissertation negatively affects the clarity of presentation and makes the reading less comfortable.
7. The dissertation is edited mostly carefully, with a small number of typos and other mistakes. However, the language, and particularly clarity of some statements and even entire paragraphs, raises questions, cf. the section "Specific remarks" below.

3.2. Specific remarks

The following list of remarks was not intended to be complete as it would exceed the scope of this review. Instead, the remarks are aimed to provide some hints to the Author when publishing the results contained in this dissertation.

1. Please provide definitions of all notions you use in your dissertation, for example, the definition of Hilbert space (page 4, p.12), weak and strong convexity (p.12), coerciveness (p.13), supercoerciveness (p.12), and so on. Please also provide clarification why is the supercoerciveness relevant.
2. Page 1: The statement "Over the last sixty years, convexity has become an essential part of the development of optimisation and variational analysis" is vague, as "convexity" has been used in this context since at least a doubly long period. Did you mean perhaps "convex analysis"?
3. Page 1: You claim that "Abstract convexity encompasses many of the above-mentioned classes of functions", but you do not specify these classes. Be sure that you mention actually relevant examples of them. For example, is invexity covered by this definition?
4. Page 2: The problem (CP) is imprecisely formulated: are X and Y *domains* or *domain spaces*? Should g(Lx) be also minimised? If yes, set the parentheses appropriately.

Please provide a clarification what is the penalty term? Do you mean here the penalty term in constraint optimisation?

5. Page 3: Is there any approach to minimise the duality gap or even achieve the zero gap on an arbitrary function?
6. Page 4: What do you mean by "*One reason may come from the fact that elementary functions are essentially nonlinear and do not belong to the dual space like gradient or subgradient in convex analysis*"?
7. Page 4: Can a subdifferential characterise a boundary points of epigraph of your objective function? Can Φ_{lsc} be applied in support of any function?
8. Page 22: What is LHS?
9. Page 23: How is the „*normal cone*" defined? Please prove that $N_p(x, C)$ is a normal cone.
10. Page 24: What is RHS?
    Please define Fréchet differentiable and ρ-weak convexity as well as Clarke and local proximal subdifferential. Please explain the difference between Fréchet differentiability and gradient in Hilbert spaces.
11. Page 30: Does the operator L exist in all vector spaces X and Y?
12. Page 30: L must be a bounded linear operator. You can refer to the original theorem of perturbation function to create the conjugate function.
13. Page 34: Is the conjugate function L^* the same as adjoint function in Fenchel's theory?
14. Page 38: How was the norm modified in the example there?
15. Page 57, Def. 8: Please explain how is it possible that both f and conjugate function f^* are less than r? Did you mean rather $r \leq f^*$?
16. Page 67: What is the motivation of defining the Lagrangian function (4.41)? The problem $\min f(x) - g(x)$ is studied as an unconstrained optimisation problem, so please explain the benefits of using the conjugate function and duality. Is there any slackness complementary (SC) condition satisfied?
17. Page 69, Theorem 8: Can the concepts and results from this theorem be generalized to non-convex optimisation problems? Are there any limitations or assumptions inherent in this theorem that restrict its applicability?
18. Page 87: The formula for $(J_\gamma)^{-1}$ is awkward and unclear. Are you able to propose any tools to compute the duality map J and its inverse J^{-1}?
19. Page 105: What is the difference between modified proximal and FB method? Can you comment on applying the FB method here?
20. Page 170: "We demonstrate the performance of Algorithm 5" or Alg.7?

3.3. Remarks to references

a) Has the manuscript „Proximal Algorithms for a class of abstract convex functions" co-authored by the Supervisors been submitted to any journal? Are the results of the review already known?

b) Please cite authors in a uniform way: for example "Stefan Rolewicz" is once quoted with the full first name and the other time with the initial only. Similarly, once you cite "Alex Rubinov" and another time as "Alexander" ([81]-[82]).

c) Do not use the abbreviation “et al.” in the list of references unless the number of authors exceeds at least 6, for example “Ewa Bednarczuk et al.”, (2023), should read: “Ewa Bednarczuk; Giovanni Bruccola; Gabriele Scrivanti; and The Hung Tran”, (2023); similarly “Bui et al.” (2022).

d) While the list of references is generally complete, nevertheless an overview of results and references to some important positions on abstract convexity are missing, for example:

Yao C.L., Li S.J.: Conjugate duality for constrained optimisation via image space analysis and abstract convexity (2019). *Optimization Letters*, 13 (2), pp. 349 - 366, DOI: 10.1007/s11590-018-1365-4 (and more papers of the same authors).

Mohebi H., Samet M.: Abstract convexity of topical functions (2014). *Journal of Global Optimization*, 58 (2), pp. 365 - 375, DOI: 10.1007/s10898-013-0037-2 (and more papers of these authors).

IV. Summary

In my opinion, the doctoral dissertation submitted by Mr. The Hung Tran satisfies the requirements of the Law on Higher Education and Science, Article 187 (Journal of Laws of July 20, 2018, item 1668, with later amendments) for awarding him the philosophy doctor degree in the scientific discipline Automatic Control, Electronics, Electrical Engineering, and Space Technologies.

The significance of the obtained results has been confirmed by two peer-reviewed articles co-authored with the Supervisor and two submitted preprints written jointly with the Supervisor and with the Co-Supervisor. The proposed optimality conditions and optimisation algorithms have been verified through computational experiments and may serve as a basis for implementations oriented towards specific applications, as evidenced, among other things, by the image processing example in Section 6.4 (Fig. 6.11).

Therefore, I am requesting the Scientific Council of the Systems Research Institute of the Polish Academy of Sciences in Warsaw, to proceed to further stages of the procedure for awarding Mr. Tran the doctoral degree, in particular to admit the doctoral dissertation submitted by Mr. The Hung Tran to public defense.

Kraków, October 16, 2024

(Prof. dr hab. inż. Andrzej M.J. Skulimowski)