

Abstract

This dissertation presents a comprehensive study of optimal control and shape optimization problems in elastic systems, combining rigorous theoretical developments with efficient numerical methods. The study covers a range of topics including linear and nonlinear elasticity, boundary contact problems, and control-driven structural design, aiming to build a unified framework that connects analysis with computation.

The study begins with the formulation of the optimal boundary control problem in linear elasticity, where the traction force is considered as the control variable. The problem is transformed into a system of state equations, adjoint equations, and optimality conditions by minimizing a cost function that includes displacement tracking and control energy. The dynamic control problem can be approximated by its static counterpart using the Turnpike property, which asserts that the optimal trajectory tends to a steady-state solution in the long term, significantly reducing the computational burden.

For shape and topology optimization, this dissertation uses shape derivative analysis combined with velocity field methods to guide the structural evolution. The gradient flow formulation transforms the optimization process into a system of time-dependent partial differential equations, which enhances numerical stability and convergence analysis. The method is shown to be very effective in flexibility minimization, shape reconstruction, and inverse problems, outperforming the classical level set method in terms of interpretability, theoretical rigor, and algorithmic control.

In this study, a discontinuous Galerkin (DG) method is used to solve the level set evolution equations when dealing with nonlinear elastic problems. The DG framework enables accurate and robust simulation of the optimization process, thus facilitating effective optimization under challenging mechanical conditions. Numerical experiments confirm the practical applicability of the proposed method.

The contact problem is solved by two approaches. The first approach uses phase field regularization to continuously model the contact interface, enabling sensitivity analysis and smooth structural evolution. The second approach uses topological derivatives to evaluate the impact of internal domain perturbations, especially in regions far from the contact boundary, combined with a phase field approach for optimization. Together, these methods form a flexible and powerful contact-aware shape and topology optimization toolkit.

The shape optimization problem is further analyzed using a gradient flow formulation in the context of a static contact problem, where elastic energy and Kohn Vogelius

Functionals are used as shape functions. Although the associated Hessian matrix is only directional due to non-smoothness, the tapered derivative framework provides a viable foundation. Numerical results show that the gradient flow system exhibits stable convergence despite theoretical challenges.

In summary, this dissertation provides a unified perspective for the control and optimization of elastic systems. It not only contributes important theoretical insights into shape sensitivity, gradient flow, and control mechanisms, but also provides robust and efficient numerical schemes. These findings lay the foundation for future dynamic optimization, multi-physics coupling, and further exploration of Turnpike phenomena in elastic systems.

Key words: Shape optimization, Optimal control, Topological derivative, Gradient flow, Phase field method